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THE ECLECTIC

SCHOOL GEOMETRY

A REVISION OF EVANS'S SCHOOL GEOMETRY

BY

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PREFACE.

THE study of Geometry trains the eye and the hand in drawing its many figures. It calls into action the reasoning faculty in mastering the demonstrations, and especially in inventing demonstrations and solutions for the theorems and problems which are left for the exercise and development of the pupil's own skill. This work demands earnest effort and the power of continuous attention. It is, hence, an excellent stimulus to those very useful traits.

In order that pupils should receive a high degree of benefit from the study, they should be incited to self-effort. The pleasure arising from victory in an attempt at original demonstration will be a powerful stimulus.

In this book, there is much work for the student besides the memorizing of definitions and axioms, and finding the road through an argument to the conclusion, with a guide constantly leading him. Sometimes the path is blazed, and the student must find his way from one mark to the next. Sometimes he must prove himself a woodsman by determining his own direction and selecting his own mode of travel. In practical life, problems refuse to confront us in a series, graded according to difficulty; and promiscuous examples may be so carefully graded as to defeat their most important end. They should, however, be fairly based on preceding principles.

When original work is called for, if there is a general failure on the part of the class, the best plan would be to pass it for the time. Very probably they will have better success with some that follow it. Let it stand as a challenge to renewed efforts.

Pupils can not be timed in this original work. It has very little resemblance to the memorizing of a poem, the mastery of a chapter of history, or the translation of a page of Cæsar.

When a theorem is stated and its demonstration is asked of the class, the pupil who is likely to reach the end in triumph, is he who draws the figure, and, sitting, walking, or standing *alone*, gives it earnest thought, seeking for known principles which he may link together, and lead to the desired conclusion. If he fail at the first attempt, he comes back with double zeal after an interval of rest. In this persistent effort, he reviews much of his stock of principles, freshens them for use, and, perchance, adds to their number.

While a text-book is an almost indispensable aid to the teacher, it should not do his work for him. He should select theorems and problems from other sources, devise new illustrations of truths discussed and things defined, and constantly put his pupils to the proof.

The *maximum* of work by the pupil, with its supplementary *minimum* by the teacher, is perhaps the true maxim.

This treatise has been prepared mainly to meet the wants of teachers and pupils in high schools and normal schools.

By omitting the Exercises, a brief course can be had, but sufficient to prepare for the study of trigonometry and surveying. We would, however, urge that the Exercises be carefully studied.

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Book II.

SOLID GEOMETRY.

Book III.

INTRODUCTION.

STUDENTS of Geometry should be led to see that while we can not tell what *space* is, all bodies, the largest and the smallest, exist in space.

We can not conceive of any boundaries, or limits, to space. We look into the sky on a clear night, and think of what astronomers tell us about the distances to a few of the stars. But these are not lights placed upon the outer wall of creation. We can not avoid believing that space reaches on and on, and that there is no end to it.

Any object that we see, as a house, field, lake, hill, book, fence, tree, is called a *body*.

Every body has size and shape and position.

Size is the result of *extension*.

The lot of ground upon which your house stands, extends, or stretches, so many feet along the street, and so many feet back from the street. These distances are found by measuring them.

The word *dimension* is from a Latin word which means to measure. The length and breadth of the lot are its dimensions.

By the use of these two dimensions, we are able to find out what is the size of the lot, the amount of ground in it. However, we do this *indirectly*, and not by laying a square foot or rod down upon the land as many times

as possible so as not to overlap, and counting the number of times.

Knowing the two dimensions already named, suppose we wish to ascertain the distance from the front left-hand corner to the rear right-hand corner; geometry teaches us how to learn this without going upon the land at all.

With a little more knowledge of geometry, we can measure distances over rivers which we can not cross, and find the height of steeples which we can not climb.

When we learn the shape and the dimensions of a water-tank, we can calculate how many gallons it will hold, though there is not a drop in it. If it were full, and if we should empty it, counting the gallons, as the farmer counts his bushels of grain from the threshing-machine, there would be no mathematical science about the operation; but when we do this by the use of certain lines, we employ Geometry. This science teaches the manner of such operations, and the reasons underlying them; and we are led to see the propriety of the following definition:

Geometry is the science which treats of the properties of figures, their construction, and the indirect measurement of their extension.

Geometry deals with lines, surfaces, and solids.

We may place before us that object which we learned in arithmetic to call a cube. It is a body, a figure, a solid.

The *space* which the cube occupies is also called a cube. It is a geometric solid.

The cube has, as we see, three dimensions, and they are called length, breadth, and thickness.

Now we can conceive of, or imagine, the thickness growing less and less until it can no longer fix the attention.

We can think of the magnitude which remains as having but two dimensions.

It is a **surface,** and is the same as one of the original six faces of the cube.

We may next imagine what would result if the breadth should do as the thickness did; and with the eye of reason we can see a **line.** It is the same as one of the edges of the cube.

If the line should shorten till the eye can detect no length, the result would be a **point.**

The epithet *imaginary* is used as definitive of geometric lines, denoting that they are visible only to the imagining, the image-making power of the mind.

NOTE TO TEACHERS.—The word is, however, on the ear of the pupil, a symbol for something so nearly synonymous with *untrue* that it is well for the teacher to give reality to the term by abundant illustration. The shortest distance from A to B is real, "actually being or existing." The plane of this page extended any distance into space is real. In short, all the air-drawn lines and surfaces of geometry,—the axis of the earth, which the pupil has learned to call imaginary, the equator, the tropics,—are as real as the Amazon River, or the great plains of Siberia.

The figures which we draw upon paper, or blackboard, are but rude, imperfect signs of the perfect figures, which exist only in the mind; though, for all geometric purposes, we may regard the symbol as the thing signified.

GEOMETRY.

SECTION I.—DEFINITIONS.

1. The term **definition** denotes a boundary. It is such a description of a thing as will distinguish it from all other things.

2. A **line** is that magnitude which has length, without breadth; or, it is the path of a point.

3. A **surface** is that magnitude which has length and breadth, without thickness; or, it is the path of a line.

4. A **point** is simply position, as the end of a line, or the crossing of two lines.

5. A **straight line** is such a line that any part will, however placed, coincide with any other part, if its extremities are made to fall on that other part; or, it is the path of a point moving without change of direction.

6. A **curve** is a line which continually changes its direction.

7. A **plane** surface is one such that if any two of its points be connected by a straight line, this line will lie wholly within the surface. Thus, the top of a desk, the face of the blackboard, or of a pane of glass, are usually planes.

8. A **solid** is that magnitude which has length, breadth, and thickness; or, it is the path of a plane.

9. A geometric **figure** is a combination of points, lines, surfaces, or solids.

Notice in what manner a line must move to describe a plane, and how a plane must move to create a solid. Notice, also, that in the figures treated of in Geometry, the parts are combined, not at random, but according to some clearly defined law.

SECTION II.—TERMS AND SIGNS.

1. Certain primary propositions, whose truth is obvious, and which are therefore incapable of proof, are called **axioms.**

2. Certain other truths, which are deduced from axioms and definitions by a train of reasoning, are called **theorems.**

3. A statement of something required to be done, or determined, is called a **problem.**

4. A **demonstration** is the proof of a theorem. It consists of the general enunciation of the truth; the special statement, involving a description of the figure; and a linking together of data and conclusions from which the final inference is drawn.

It is well to notice that not the general theorem, but only one instance, is actually demonstrated; but that the mode of doing it will apply to every instance which comes under the original conditions. As an illustration, see Theorem XI., Book 1. The point D is selected at will within the triangle, yet, when chosen, it is a fixed position. But the reasoning would be the same if any other point within the triangle were taken.

5. **Construction** lines are additional lines drawn to aid in the demonstration.

6. A **Corollary** is a consequence, obvious, or readily deduced from something which precedes.

7. A **scholium** is a remark.

8. A **postulate** grants the solution of a self-evident problem; as, that a straight line may be prolonged, or that a straight line may be drawn between any two points.

9. An **hypothesis** denotes the supposition made, or the conditions given, in any proposition.

Scholium.—In Theorem XI., just referred to, the condition, that the two straight lines be drawn from the ends of one side of a triangle to a point within the triangle, is the hypothesis.

10. When the conclusion of one proposition is taken as the hypothesis of another, each proposition is the **converse** of the other.

Sch. 1.—For an illustration, turn to Theorem IV. The hypothesis is that a line intersect two parallels. The conclusion relates to the sum of certain angles. We shall have the converse of this proposition if we assume that the sum of two certain angles is two right angles, and conclude that the two lines in question are parallel.

Sch. 2.—The truth of a converse is not a necessary consequence of the truth of the original theorem, but requires a distinct demonstration.

11. The term **infinite**, as used in Geometry, means beyond measure; that is, either absolutely beyond limits, or beyond all appreciable limits.

12. The **ratio** of one quantity to another is the quotient obtained when the first is divided by the second, as $A : B$, or $A \div B$, or $\frac{A}{B}$.

13. A **proportion** is an equality of ratios, as $A : B = C : D$, or $\frac{A}{B} = \frac{C}{D}$.

14. $A + B$ is read A plus B.

15. $A - B$ is read A minus B.

16. $A \times B$ is read A multiplied by B.

17. $A \div B$ is read A divided by B.
18. $A < B$ is read A is less than B.
19. $A > B$ is read A is greater than B.
20. $(A + B)(A - C)$ is read the sum of A and B is multiplied by their difference.
21. A^2 is read A squared.
22. $\sqrt{A} = \sqrt[3]{B}$ is read the square root of A is equal to the cube root of B.
23. The ? when used in this book at the end of a statement, calls upon the pupil for a reason.
24. The sign $\therefore$ is read *therefore.*

SECTION III.—AXIOMS AND POSTULATES.

Some truths are known to us as soon as we comprehend the terms in which they are expressed. Others are known to us through the medium of these first truths. The former are the subject of intuition; the latter, of inference. The greatest part of our knowledge is the result of inference.

Intuitive or self-evident truths are called **axioms.**—(Sec. II., Def. 1.)

Some of them lie at the base of mathematics in general; others relate specially to magnitudes in space, and are therefore peculiar to Geometry.

Axioms.

1. Things equal to the same thing are equal to each other.
2. If equals be added to equals, the sums will be equal.
3. If equals be taken from equals, the remainders will be equal.

4. If equals be added to unequals, the sums will be unequal.

5. If equals be taken from unequals, the remainders will be unequal.

6. Doubles of the same thing are equal to each other.

7. Halves of equal things are equal to each other.

8. From one point to another only one straight line can be drawn; and that is the shortest line between them.

9. If two straight lines have two points common, they will coincide throughout their mutual extent.

10. Two magnitudes are equal if, when one is applied to the other, they exactly coincide.

Postulates.

1. A straight line may be drawn in any direction from a point.

2. A circle may be described from any center with a radius equal to any straight line.

3. A magnitude may have any form.

4. A magnitude may have any extent.

Definitions.

1. The **whole** of any magnitude is all of it.

2. A **part** is some portion less than all of it.

Cor. 1.—The whole is equal to the sum of all the parts.

Cor. 2.—The whole is greater than any of its parts.

PLANE GEOMETRY.

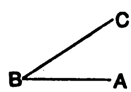
BOOK 1.

SECTION IV.—STRAIGHT LINES AND THEIR ANGLES.

DEFINITIONS.

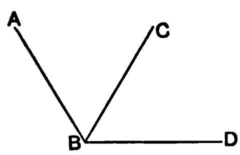
1. An **angle** is formed by two straight lines diverging from a common point. The quantity of the angle is the amount of this divergence.

The common point is called the **vertex.** The lines are the **sides** of the angle.



Thus, BA and BC are the sides, and B the vertex, of the angle ABC; or, as it may be read, the angle B.

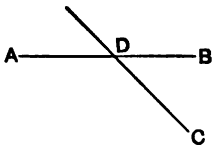
When two angles have a common vertex, and one side common, they are called **adjacent angles.**



Thus, ABC and CBD are adjacent angles.

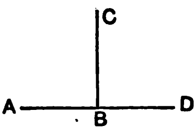
2. When the adjacent angles made by one straight line with another are equal, the angles are **right angles.**

An angle greater than a right angle is **obtuse**; an angle less than a right angle is **acute.**



Thus, ADC is obtuse; BDC is acute.

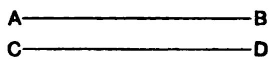
3. When one straight line makes right angles with another, either line is **perpendicular** to the other.



Thus the angles at B are right angles. AD is perpendicular to CB, and CB to AD. If the angles formed by two lines meeting are unequal, the lines are **oblique**.

4. Two straight lines are **parallel** when they have the same direction.

Thus, AB and CD are parallel.



Cor.—Parallels will never meet, however far they may be produced; for, if they were to meet, they would make an angle with each other; that is, they would differ in direction, which is contrary to the definition.

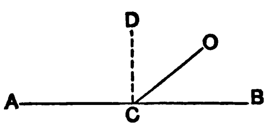
Sch.—Any line may be said to have two exactly opposite directions; as the line AB is the path of a point moving either from A to B, or from B to A.

5. Any magnitude is said to be **bisected** when it is divided into two equal parts.

Cor.—A straight line has one, and but one, point of bisection.

THEOREM I.

The two angles which one straight line makes with another, on one side of it, are together equal to two right angles.



Let the line OC move around the fixed point C so that OCB will constantly increase, and OCA as constantly decrease. In one position, as DC, these

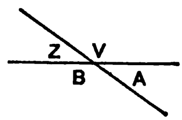
two angles must be equal. They are then right angles (Def. 2, Sec. IV.); but their sum is the same as it was in all other positions of OC. Therefore, etc. (Repeat the theorem.)

Cor. 1.—Whatever be the number of angles formed at C, on one side of a straight line, their sum is two right angles.

Cor. 2.—At a given point, C, but one perpendicular can be erected to the line AB (?). (See Def. 3.)

THEOREM II.

If two straight lines cut one another, the opposite or vertical angles are equal.



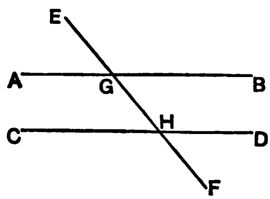
By Theo. I., $Z + V = 2$ right angles.
" " " $A + V =$ " " "
$\therefore$ (Axiom 1) $Z + V = A + V$
and (Axiom 3) $Z = A$.
Therefore, etc.

Sch.—In the same way it may be proved that B and V are equal.

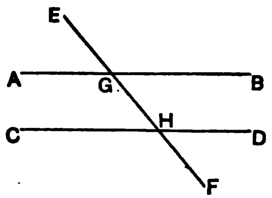
Cor.—If Z or V is a right angle, each of the other three angles about the common point is a right angle (?).

THEOREM III.

If a straight line intersect two parallels, the corresponding inner and outer angles will be equal to each other; also the alternate angles.



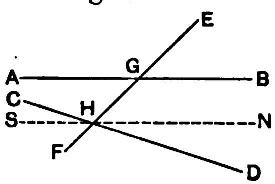
Let the straight line EF intersect the two parallels AB and CD in G and H. Then will any two corresponding inner and outer angles, as GHD and EGB, be equal. Since EF



is a straight line, EG has the same direction from G that GH has from H (Def. 5, Sec. I.); and, since HD and GB are by hypothesis parallel, HD has the same direction from H that GB has from G (Def. 4, Sec. IV.); therefore, the difference in direction between HG and HD is equal to that between GE and GB: that is, the angle GHD is equal to the angle EGB (Def. 1, Sec. IV).

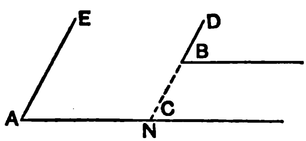
Again, any two alternate angles, as AGH and GHD, are equal, for each is equal to EGB (?).

Cor. 1.—If AB be not parallel to CD, the corresponding inner and outer angles will be unequal; also, the alternate angles.

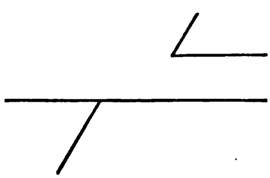


Let the student prove GHD $>$ EGB, using the construction line SN drawn parallel to AB. GHN $=$ EGB (?).

Cor. 2.—If two angles have their sides parallel, each to each, and directed the same way from the vertex, they are equal.



A and B are the given angles. Prolong the line DB to N. Ang. B $=$ Ang. C (?). Complete.

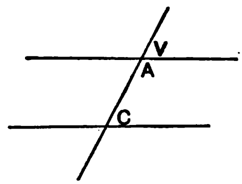


Cor. 3.—If two angles have their sides parallel, but extended in directly opposite directions from the vertex, they are equal (?).

What are corresponding outer and inner angles? Alternate angles?

THEOREM IV.

If a straight line intersect two parallels, the sum of the two inner angles on the same side will be two right angles.



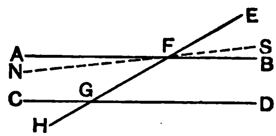
Denote a right angle by this symbol ∟.

V + A = 2 ∟'s (Theo. I).

C = V (Theo. III).

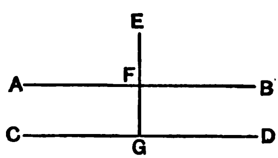
∴ C + A = 2 ∟'s.

Conversely (Def. 10, Sec. 2), if a straight line intersects two other lines, and the sum of the two interior angles on the same side is two right angles, the two lines cut by the first are parallel.



By hypothesis, BFG + FGD = 2 right angles. If AB be not parallel to CD, through the point F draw NS parallel to CD. Then (Theo. IV.) SFG + FGD = 2 right angles. ∴ BFG + FGD = SFG + FGD. ∴ BFG = SFG. This can not be true (Sec. III, Def. 2, Cor. 2), and the error arose from assuming that any line other than AB can be drawn through F parallel to CD. Therefore, AB is parallel.

Cor. 1.—If a straight line is perpendicular to one of two parallels, it is perpendicular to the other (?).



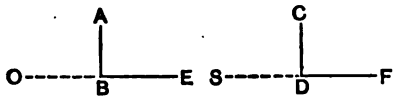
BFG + FGD = 2 ∟'s. But, by hypothesis, EF is perpendicular to AB. Therefore, EFB and EFA are right angles; also (Theo. II), BFG is a right angle.

∴ FGD is a right angle, and FG is perpendicular to CD.

Cor. 2.—Two lines perpendicular to a third are parallel (?).

THEOREM V.

All right angles are equal.



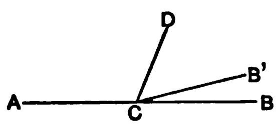
The angles ABE and CDF are any two right angles. Apply DF to BE, letting D fall upon B. DS, the prolongation of DF, will fall upon BO, the prolongation of BE (Axiom 9). DC will fall upon BA (Theo. I, Cor. 2), and the given right angles are shown to be equal (Axiom 10).

Sch. 1.—The truth of Theorem V may be inferred from Theorem I; but this proof is given as applying to any two right angles.

Sch. 2.—The sum of all the angles, which can be drawn in a plane about a given point, is represented numerically by 360°. Hence, one right angle is equal to 90°.

THEOREM VI.

If the adjacent angles, made by one straight line with two others, are together equal to two right angles, these two lines are in one straight line.



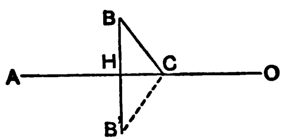
The line DC, meeting AC and BC, makes the angles ACD and DCB, whose sum, by hypothesis, is two right angles. Then, are AC and CB in one straight line. If CB is not the prolongation of AC, some other line, as CB′, must be. If ACB′ is a straight line, ACD + DCB′ = 2 L's. ∴ ACD + DCB′ = ACD + DCB.

Subtracting ACD (Axiom 3), and DCB′ = DCB; that is, the whole is equal to one of its parts. This is un-

true (Sec. III, Def. 2, Cor. 2), and the error arose from taking some other line than CB as the prolongation of AC. This error will not appear when we assume CB as the prolongation of AC; therefore, it is such prolongation.

THEOREM VII.

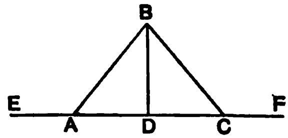
If a perpendicular and oblique lines are drawn from the same point to a given line, the perpendicular is shorter than any oblique line.



BH is a perpendicular and BC an oblique line from B to AO. Then, BH is shorter than BC. Continue BH till HB′ shall equal BH; and draw B′C. Rotate the upper part of the figure upon AO as an axis. BH will at length fall upon HB′, as the angles at H are right angles. The point B will fall upon B′, since HB′ = BH. And the line BC will coincide with and be equal to B′C (Axiom 8). Now, the broken line BCB′ is longer than the straight line BB′, joining the same points (Axiom 8). Therefore the perpendicular BH, which is half of BHB′, is shorter than BC, which is half of BCB′ (Axiom 7); that is, a perpendicular is shorter than an oblique line, if both are drawn to a line from any point outside that line.

THEOREM VIII.

If two oblique lines drawn from a point meet the given line at equal distances from the foot of the perpendicular, they are equal.

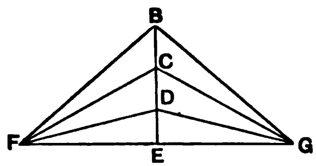


The oblique lines BA and BC meet the line EF at the points A and C, so that AD = CD.

Rotate the left side of the figure upon the perpendicular BD as an axis. ED will fall upon DF (Theo. V). The point A will fall upon C. Therefore (Axiom 8), AB = CB.

THEOREM IX.

Every point in a line which is perpendicular to another line at its center, is equally distant from the two ends of the line.



By construction, FE = GE, and BE is perpendicular to FG.

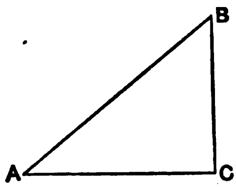
If by rotation about BE as an axis the point F be made to fall upon G, then will FB = GB (Axiom 8); FC = GC, and FD = GD.

SECTION V.—TRIANGLES.

DEFINITIONS.

1. **A plane figure** is a portion of a plane bounded on all sides by lines.
2. **A polygon** is a plane figure bounded by straight lines.
3. **A triangle** is a polygon of three sides.

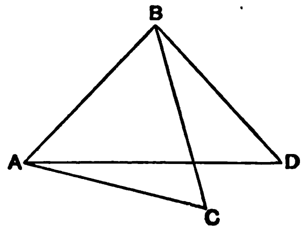
If one of its angles is a right angle, the triangle is called a **right-angled** triangle; and the side opposite the right angle is called the **hypotenuse.**



Thus the plane figure ABC is a right-angled triangle. AB is the hypotenuse. CA and CB are sometimes called the **legs**.

An **isosceles** triangle has two equal sides.

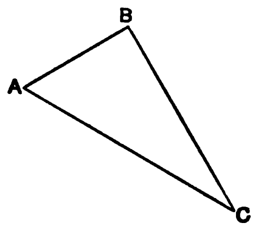
An **equilateral** triangle has three equal sides.



Thus, ABD is isosceles, and ABC is equilateral. An equilateral triangle is always isosceles.

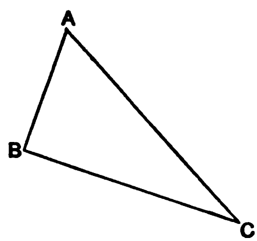
A **scalene** triangle has no two sides equal.

Can one side of a triangle be equal to the sum of the other two? Can it be greater?



Thus, the triangle ABC is scalene.

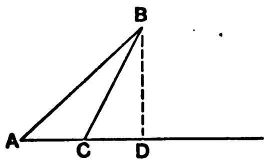
4. Any side of a triangle may be considered as its **base,** or that upon which it is supposed to rest, and the opposite angle as the **vertex;** but in an isosceles triangle the **base** is the side that is not equal to any other side.



Thus, with reference to BC as the base, A is the vertex. When we regard BA as the base, C is the vertex; similarly with AC and B.

When two sides of a triangle have been named in a demonstration, the third is often called the **base.**

The **altitude** of a triangle is the perpendicular let fall from the vertex on the base, or the base produced.



Thus, BD is the altitude of the triangle ABC.

5. A polygon is **equiangular** when all its angles are equal.

6. Two polygons are called **mutually** equilateral, or **mutually** equiangular, if the sides or angles of the one are equal to the sides or angles of the other, each to each, taken in the same order.

7. When the sum of two angles is two right angles, or 180°, they are termed **supplementary.** When their sum is one right angle, or 90°, they are termed **complementary.**

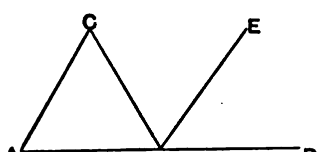
Cor. 1.—The supplements of equal angles are equal?

Cor. 2.—The complements of equal angles are equal?

8. **An exterior** angle is formed by a side of a polygon, and an adjacent side produced. It is the supplement of the adjacent interior angle (Theo. 1).

THEOREM X.

The three interior angles of a triangle are together equal to two right angles.

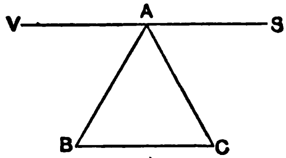


The construction line BE is drawn parallel to AC.

EBD = CAB (Theo. III.)
CBE = ACB " "
CBA = CBA

EBD + CBE + CBA = CAB + ACB + CBA (Axiom 2).

The first member of this equation is equal to two right angles (Theo. I, Cor. 1). Therefore the second member, which is the sum of the three interior angles, is equal to two right angles.



NOTE.—Let the pupils give the demonstration, using this figure.

Ques. 1.—To what is an exterior angle, as CBD, equal?

Ques. 2.—How many right angles may a triangle have?

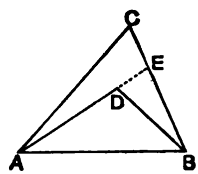
Ques. 3.—There being 90° in a right angle, how many degrees are there in an angle of an equiangular triangle?

Cor. 1.—The two acute angles of a right-angled triangle are complementary.

Cor. 2.—Any angle of a triangle is the supplement of the sum of the other two angles.

THEOREM XI.

If two straight lines be drawn from the extremities of one side of a triangle to a point within, their sum will be less than that of the other two sides of the triangle.



The point is D. The line AD is extended to E.

Now, $DB < DE + EB$?

$\therefore \quad AD + DB < AE + EB$ (?).

$AE < AC + CE$?

$\therefore \quad AD + DB < AC + CE + EB$

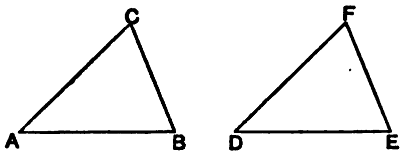
or $\quad AD + DB < AC + CB$.

NOTE.—Have pupils give demonstration, prolonging BD instead of AD.

THEOREM XII.

If two triangles have two sides and the included angle of the one, respectively, equal to two sides and the included angle of the other, they are equal throughout.

The triangles ACB and DFE have AC, the ang. A, and AB respectively equal to DF, the ang. D, and DE.



Conceive the triangle ACB to move to the right till AB coincides with DE. Then, as the ang. A = ang. D, AC will fall upon DF. As the points C and B fall upon F and E, CB must be the same as FE (Axiom 10).

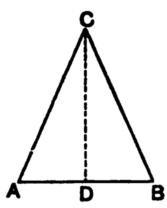
Cor.—In equal triangles, note the relative position of equal sides and angles.

Sch. 1.—Axiom 10 implies a definition of geometric **equality,** and this theorem illustrates it.

Sch. 2.—It would be well to have pupils demonstrate Theorems VIII and IX, as corollaries under Theorem XII.

THEOREM XIII.

The angles at the base of an isosceles triangle are equal.



The triangle ACB is isosceles. Bisect the angle C by CD. The side AC = BC by definition. CD is common, and the angles ACD and BCD are equal by construction. Therefore (Theo. XII) the triangles ACD and BCD are equal in all their parts, and the ang. A = ang. B.

Cor. 1.—CD is perpendicular to AB (?).

Cor. 2.—AD = DB (?).

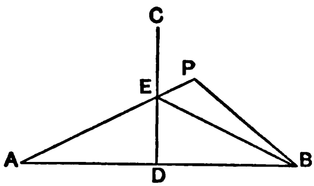
Cor. 3.—Every point in the perpendicular CD is equally distant from the ends of AB (Theo. XII).

Cor. 4.—Every point without the perpendicular CD is unequally distant from the ends of AB.

AE = EB. AE + EP = EB + EP.

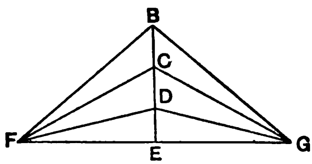
But EB + EP > PB (?).

Complete demonstration.



Cor. 5.—A line having two points equally distant from the extremities of another line, is perpendicular to it at its center.

The points B and C are equally distant from F and G. If a perpendicular to FG, at its center E, be erected, it must pass through C and B (Cor. 4), and must coincide with BE (Axiom 9). Therefore, BE is perpendicular to FG.

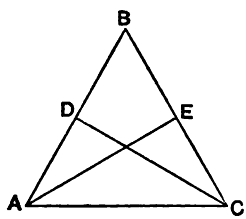


Sch.—This (Cor. 5) is the converse of Theo. IX.

THEOREM XIV.

Lines joining the middle points of the sides of an isosceles triangle to the opposite extremities of the base are equal.

ABC is an isosceles triangle, of which AC is the base. By construction, AD = DB and CE = EB. Therefore (Axiom 7), AD = EC.

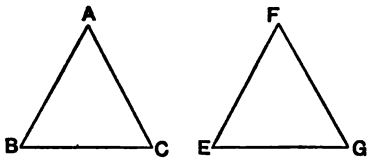


Then, in the triangles, ADC and AEC, the line AC is common, AD = EC, the included angles A and C are equal (Theo. XIII). ∴ the triangles ADC and AEC are equal (Theo. XII), and DC = AE.

Cor.—Place the letter N at the intersection of AE and DC, and prove the triangles AND and CNE equal.

THEOREM XV.

If a triangle have two equal angles it is isosceles.



In the triangle ABC, the angles B and C are equal. It is required to prove the triangle isosceles.

Draw the triangle FEG equal to ABC. Conceive the paper so folded that B shall fall upon G, and C shall fall upon E.

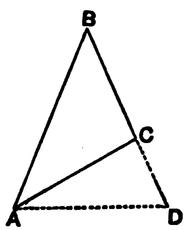
Then, since ang. B = ang. G, BA will fall upon GF. And as ang. C = ang. E, CA will fall upon EF. Therefore, the point A is found in both EF and GF, and must be at their intersection F. ∴ AB = FG(?). But, by construction, AC = FG. ∴ AB = AC (Ax. 1), and the triangle ABC is isosceles.

Cor.—An equiangular triangle is also equilateral.

Taking either angle as the vertex, the other two angles are at the base (Theo. XIII).

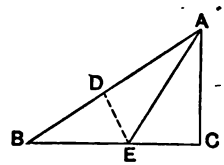
THEOREM XVI.

The greater of two unequal sides in a triangle has the greater angle opposite to it.



The side AB > BC. ∴ ang. C, opposite AB, is greater than ang. A, opposite BC.

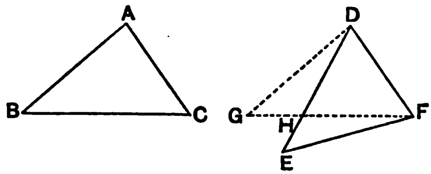
Produce BC till BD equals AB, and draw AD. Then the ang. D is equal to ang. BAD (Theo. XIII). Now the angle ACB is an exterior angle (Def. 8, Sec. V), and is equal to CDA plus CAD (Theo. I and Theo. X). ∴ BCA > D, and as D > BAC(?) ∴ BCA > BAC.



Use this figure. The line AE bisects ang. A, and AD = AC. Triangles ADE and ACE are equal (Theo. XII). ∴ ang. ADE = ang. ACE. But ADE > ang. B (?).

THEOREM XVII.

If two triangles have two sides of the one equal to two sides of the other, but the included angles unequal, the base of that which has the greater angle is greater than the base of the other.



Let the two triangles ABC, DEF, have the two sides AB, AC, respectively equal to the two sides DE, DF, but the angle BAC greater than the angle EDF. It is to be proved that the base BC is greater than the base EF.

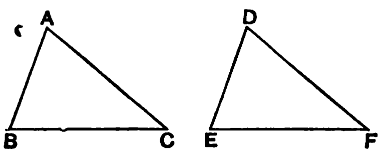
Because the angle EDF is, by hypothesis, less than the angle BAC, the sum of the angles E and F must be greater than the sum of the angles B and C (Theo. X); hence, either E is greater than B, or F is greater than C. Suppose, then, that F is greater than C. On the adjacent side DF, describe the triangle DGF equal to the triangle ABC, and so placed that DG shall be the side equal to AB. Now, the sum of DH and HG is greater than DG (Axiom 8); and the sum of EH and HF is greater than EF. Therefore, the sum of the entire lines DE and GF is greater than the sum of DG and EF. Taking away from these unequals the equals DE and DG, we have the remainder GF greater than the remainder EF. But GF is equal to BC by construction.

Therefore, BC is greater than EF. Hence, if two triangles, etc.

Cor.—Conversely, if in two triangles, ABC and DEF, we have AB $=$ DE, AC $=$ DF, and BC $>$ EF; then, A $>$ D. If A $=$ D what? If A $<$ D what?

THEOREM XVIII.

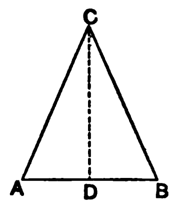
If two triangles have two angles, and the included side of the one equal to two angles and the included side of the other, each to each, they are equal throughout.



Let ABC, DEF, be two triangles having the angles A and B respectively equal to the angles D and E, and the included side AB equal to the included side DE. It is to be proved that these triangles are equal throughout.

If the triangle ABC be applied to the triangle DEF, so that the side AB shall coincide with its equal DE, then because the angle A is equal to the angle D, the side AC will fall on the side DF, and the point C will be found somewhere in the line DF; also, because the angle B is equal to the angle E, the side BC will fall on the side EF, and the point C will be found somewhere in the line EF; consequently the point C, being in both the lines DF and EF, must be at their point of intersection F. Hence, the two triangles exactly coincide, and are equal throughout (Axiom 10): that is, the side AC is equal to the side DF, the side BC to the side EF, the angle C to the angle F, and the triangle ABC as a whole to the triangle DEF as a whole.

Cor. 1.—Prove the corollary under Theo. XIV as an application of this theorem.



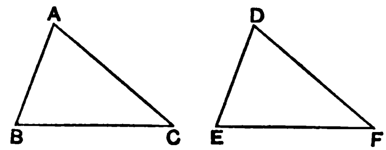
Cor. 2.—Demonstrate Theo. XV with this figure.

The angle A is equal to the angle B, by hypothesis; the angle ACD is equal to the angle BCD, by contruction; and the angle CDA is equal to the angle CDB (?).

The side CD is common.

TREOREM XIX.

Two triangles which are mutually equilateral are also mutually equiangular.



Let the two triangles, ABC, DEF, have the three sides AB, BC, CA, respectively equal to the three sides DE, EF, FD. Then, also, will the angles A, B, and C be respectively equal to the angles D, E, and F.

If the angle A is not equal to the angle D, it must be either greater or less. But it can not be greater than D, for then the base BC would be greater than the base EF (Theo. XVII), which is contrary to the hypothesis. Neither can it be less than D; for, in that case, BC would be less than EF, which is also contrary to the hypothesis. Hence, A must be equal to D; and, in the same manner, it may be proved that B is equal to E, and C to F.

Therefore, two triangles, etc.

Cor. 1.—If two triangles are mutually equilateral, they are equal throughout.

Cor. 2.—Mutually equiangular triangles may not be equal.

EXERCISES.

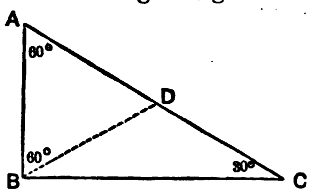
1. Prove that if two angles of one triangle are equal to two angles of another triangle, the third angles are also equal.

2. Prove that each angle of an equilateral triangle is equal to two thirds of a right angle.

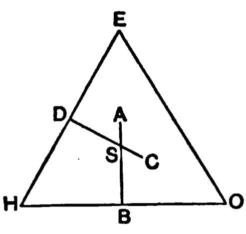
3. Prove that, from a given point without a straight line, only one perpendicular to that line can be drawn.

4. Draw the Fig. for Theo. XI, and prove that the angle D > C.

5. In a right-angled triangle, if one of the acute angles is one third of a right angle, the side opposite is one half of the hypotenuse. Draw BD, making ABD = BAD. Then the triangle ABD will be equilateral (?), and the triangle DBC will be isosceles (?). ∴ AB = AD = BD = DC. ∴ $AB = \frac{AC}{2}$.



6. The perpendiculars to the sides of a triangle at their centers meet at one point.



The line AB being drawn perpendicular to the center of HO, every point of it is equally distant from H and O (Theo. XIII, Cor. 3). Similarly, every point of CD is equally distant from H and E. The intersection of CD and AB, or S, is equally distant from E and O, since ES and OS each equals HS. Therefore, a perpendicular from the center of EO will pass through S (Theo. XIII, Cor. 4).

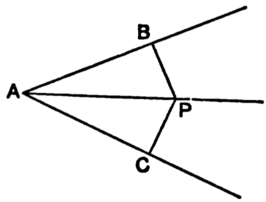
7. Every point in the line which bisects an angle is equally distant from the sides of the angle.

From the point P drop the perpendiculars PB and PC. It is required to prove PB = PC.

By hypothesis, ang. BAP = ang. PAC.

By definition, ang. PBA = ang. PCA.

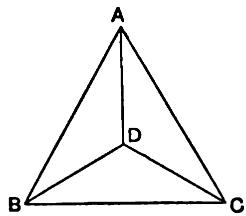
Complete demonstration.



8. The bisectors of the three angles of a triangle pass through the same point.

The lines AD and BD, since they are not parallel, must intersect at some point, say D.

The point D is therefore equally distant from AC and BC (?), and therefore the bisector CD must pass through it.



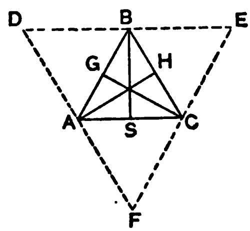
Cor.—The perpendiculars from D to the three sides of the triangle are equal.

9. The perpendiculars from the three vertices of a triangle to the opposite sides pass through the same point.

Through the vertices A, B, and C, draw lines parallel to the opposite sides. The triangles DBA, BEC, and ACF are each equal to ABC (?) (Theos. III and XVIII).

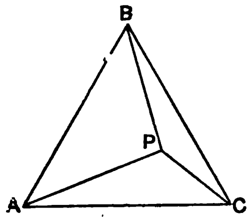
∴ DE = 2 AC, FE = 2 AB, and DF = 2 BC (?).

Since BS is perpendicular to DE at its center, etc. Complete.



10. If, from a point within a triangle, lines be drawn to the three vertices, their sum will be less than the

sum of the three sides, and greater than the half-sum.



AP + PC < AB + BC (Theo. XI).

AP + PB < AC + BC.

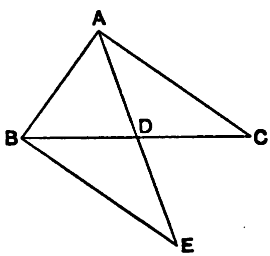
Complete, and draw the first conclusion.

AP + PC > AC (Axiom 8).

AP + PB > AB.

Complete, and draw the second conclusion.

11. The line which joins the vertex of a triangle to the center of the base is less than half the sum of the sides.



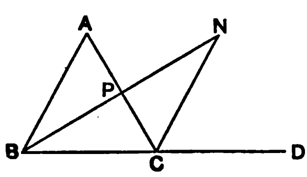
The line AD, drawn to the center of BC, is prolonged till DE is equal to AD.

AC = BE (Theo. XII).

AE < AB + BE.

Complete the proof.

12. The point P is the center of the side AC. BP is drawn and continued till PN = BP.



CN is parallel to BA (?).

Compare the triangles APB and NPC, and see Cor. 1, Theo. III.

13. The bisectors of two adjacent supplementary angles are at right angles to each other.

14. The angles of a triangle are in the ratios of 1, 3, and 5. What is the size of each angle?

SECTION VI.—QUADRILATERALS.

DEFINITIONS.

1. A **quadrilateral** is a polygon of four sides.
2. A **parallelogram** is a quadrilateral having its opposite sides parallel.
3. A **rectangle** is a parallelogram whose angles are all right angles.
4. A **square** is a rectangle whose sides are all equal.

Ques.—Why is a square a parallelogram? Why a quadrilateral?

5. A **trapezoid** is a quadrilateral having two opposite sides parallel.
6. A **diagonal** of a polygon is a line joining two angles not adjacent.

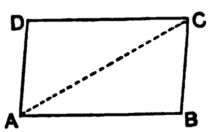
Ques.—How many diagonals may a quadrilateral have? A triangle?

7. The **area** of any plane figure is the amount of surface which it contains.
8. Two plane figures which will not coincide, but which have equal areas, are **equivalent.**

Ques.—When are two plane figures equal? (Ax. 10.)

THEOREM XX.

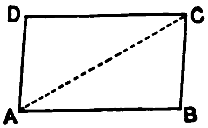
The opposite sides and angles of a parallelogram are equal to each other.



The figure ABCD is a parallelogram. Draw the diagonal AC.

How many pairs of parallels does it cut?

Why is the angle DCA equal to the angle CAB? Why is CAD equal to ACB?.



The triangles ABC and ADC are equal (Theo. XVIII).

AD and BC are opposite equal angles; likewise DC and AB.

Therefore, the opposite sides are equal.

Ang. ACB = ang. CAD.
Ang. ACD = ang. CAB.
Adding, DCB = DAB.
Prove ang. D = ang. B.

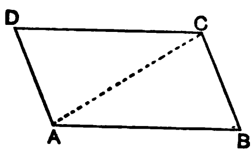
Cor. 1.—A diagonal of a parallelogram divides it into two equal triangles?

Cor. 2.—Two parallels are everywhere equally distant.

Def.—Any side of a parallelogram may be taken as its base; and a perpendicular let fall from any point in the opposite side on the base, or the base produced, is called the **altitude** of the parallelogram.

THEOREM XXI.

If a quadrilateral has two of its sides equal and parallel, it is a parallelogram.



Let the quadrilateral ABCD have the sides AB, DC equal and parallel. Then the figure is a parallelogram.

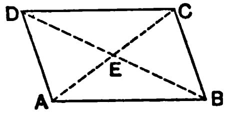
Draw the diagonal AC, and the alternate angles DCA and BAC are equal (?). Then the triangles CDA and CBA are equal (Theo. XII) in all their parts; and, as ang. ACB = ang. CAD, the lines AD and BC are parallel (?).

Therefore (Def. 2, Sec. VI), the quadrilateral ABCD is a parallelogram.

Ques.—Is a quadrilateral containing two right angles necessarily a parallelogram? Three?

THEOREM XXII.

The diagonals of a parallelogram bisect each other.



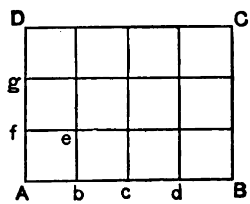
Let ABCD be a parallelogram. It is to be proved that its diagonals, AC, BD, bisect each other in E.

In the triangles AEB, DEC, since the angle ABD is equal to its alternate angle BDC (Theo. III), and the angle BAC equal to its alternate angle ACD, and the included side AB equal to the included side DC (Theo. XX), it follows that the two triangles are equal (Theo. XVIII); consequently, the side AE is equal to the side EC, and BE to ED (Cor., Theo. XII); that is, the two diagonals bisect each other in E.

Therefore, the diagonals, etc.

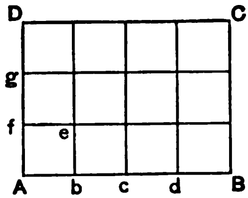
THEOREM XXIII.

The area of a rectangle is equal to the product of its base by its altitude.



Let ABCD be a rectangle. It is to be proved that its area is equal to the product of its base AB by its altitude AD.

Let AB be divided into a certain number of equal parts, A*b*, *bc*, etc., taken as the units of length; also, let AD be divided into a certain number of the same units, A*f*, *fg*., etc. From *b*, *c*, etc., draw straight lines parallel to AD, and from *f*, etc., draw straight lines parallel to AB. Now, it is evident that the whole rectangle is divided into small squares (Cor. 2, Theo. XX), each equal to A*bef;* which may be taken as the unit of area. Of these equal squares, there are as many in the tier next to



AB as there are units of length in AB; and there are as many equal tiers in the whole figure as there are units of length in AD. Therefore, the whole number of units of area in ABCD is equal to the number of linear units in AB multiplied by the number of linear units in AD.

Hence, the area of a rectangle, etc.

Sch. 1.—This measurement gives the numerical area, as we multiply the *length* of the base by the *length* of the altitude. The above form of statement is used for brevity.

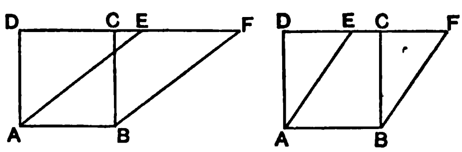
Sch. 2.—If AB and AD are incommensurable, that is, if no unit can be found into which they can both be divided without leaving a remainder in one of them, the theorem will still hold true; for if the unit be taken smaller and smaller, the remainder can be made less than any assignable quantity.

When the linear unit is one inch, the unit of area is a square inch; when the linear unit is one foot, the unit of area is a square foot, etc.

Cor.—Since the base and altitude of a square are equal (Def. 4, Sec. VI), its area may be found by multiplying one side into itself.

THEOREM XXIV.

The area of any parallelogram is equal to the area of a rectangle having the same base and altitude.



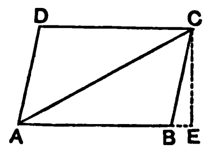
Let ABCD be a rectangle, and ABFE any parallelogram, on the same base

AB, and of the same altitude, namely, the perpendicular distance between the parallels AB, DF. Then will ABFE be equivalent to ABCD.

Since AB and DC are opposite sides of a parallelogram, they are equal (Theo. XX); and, for the same reason, AB and EF are equal; therefore, DC is equal to EF (Ax. 1). Taking away each of these in turn from the whole line DF, we have the remainder DE equal to the remainder CF. But DA is equal to CB (Theo. XX), and the included angle ADE is equal to the included angle BCF (Theorem III); therefore, the triangles ADE, BCF are equal (Theo. XII); and hence, if each of them be taken away in turn from the whole figure ABFD, the remainder ABFE will be equivalent to the remainder ABCD.

Therefore, the area of any parallelogram, etc.

Cor. 1.—The area of any parallelogram is equal to the product of its base by its altitude.



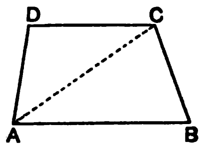
Cor. 2.—Since any triangle, as ABC, is half of a parallelogram, ABCD (Cor. 1, Theo. XX), on the same base, AB, and having the same altitude, CE (Cor. 2, Theo. XX), it follows that *the area of a triangle is equal to half the product of its base by its altitude.*

Cor. 3.—Triangles on the same base or on equal bases, and of equal altitudes, have equal areas.

Cor. 4.—Triangles of equal areas, on the same base or on equal bases, have equal altitudes.

THEOREM XXV.

The area of a trapezoid is equal to half the product of the sum of its parallel sides by its altitude.



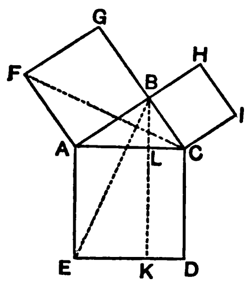
Let ABCD be a trapezoid of which AB and DC are the parallel sides. Then will its area be equal to half the product of the sum of AB and DC into its altitude, namely, the perpendicular distance between AB and DC.

Draw the diagonal AC. Now, the area of the triangle ABC (Cor. 2, Theo. XXIV) is equal to half the product of its base AB into its altitude, which is the same as the altitude of the trapezoid; again, the area of the triangle ADC is equal to half the product of its base DC into its altitude, which is also the same as the altitude of the trapezoid. Therefore, adding together the area of the two triangles, we have the area of the whole figure equal to half the product of the sum of AB and DC into the altitude.

Hence, the aréa of a trapezoid, etc.

THEOREM XXVI.

The square described on the hypotenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.



Let ABC be a triangle right-angled at B. It is to be proved that the square AEDC is equivalent to the sum of the squares ABGF and BHIC.

Join FC, BE, and draw BK parallel to AE. Also observe that, since ABC and ABG are right angles, BC and BG form one straight line.

Now, in the triangles EAB, CAF, the side EA is equal to the side CA, since they are sides of the same square, and, for the same reason, the side AB is equal

to the side AF. But the included angles EAB, CAF are also equal; for each of them is composed of a right angle and the angle CAB. Therefore, the two triangles are equal (Theo. XII).

But, since the triangle CAF and the square BAFG have the same base, AF, and the same altitude, namely, the perpendicular distance between the parallels AF, CG, the square must be double the triangle (Cors. 1 and 2, Theo. XXIV). For like reason, the parallelogram AEKL is double the triangle EAB. But the doubles of equals are equals; therefore, the parallelogram AEKL is equivalent to the square BAFG. In the same manner (by joining AI and BD), it may be shown that the parallelogram CDKL is equivalent to the square BCIH. Hence, the whole square ACDE is equivalent to the sum of the squares BAFG, BCIH.

Therefore, the square described, etc.

Cor. 1.—The hypotenuse is equal to the square root of the sum of the squares of the other two sides.

Cor. 2.—The square of either of the sides containing the right angle is equivalent to the square of the hypotenuse diminished by the square of the other side. By taking the square root of the remainder, the side itself will be found.

Cor. 3.—If two right-angled triangles have the hypotenuse and one side of the one respectively equal to the hypotenuse and one side of the other, the third sides will also be equal. For, if $H^2 - B^2 = A^2$, and $H^2 - B^2 = C^2$, $A^2 = C^2$, and $A = C$. Or thus: If two right-angled triangles have each a side equal to AB, and each a side equal to AC, make the pair equal to AB coincide; then, if the other equal pair do not coincide, we shall have two equal oblique lines from a point to a line, on one side of the perpendicular. Can this be?

EXERCISES.

1. If the side of a square is 36 inches, what is its area in square inches? What in square feet?

2. If the base of a parallelogram is 3 feet, and its altitude 4 feet and 6 inches, what is its area?

3. If the base of a triangle is 50 yards, and its altitude 20 yards, what is its area?

4. If the parallel sides of a trapezoid are 12 rods and 16 rods, and its altitude $8\frac{1}{2}$ rods, what is its area?

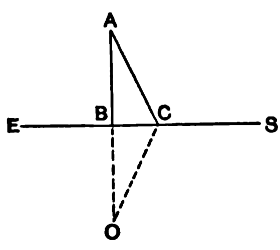
5. If the sides containing the right angle af a right-angled triangle are 3 and 4, what is the length of the hypotenuse?

6. If the hypotenuse is 10, and one of the sides 8, what is the length of the other side?

7. Prove that if a parallelogram has one right angle it is a rectangle.

8. Prove that the diagonals of a rectangle are equal to each other. (Cor. 1, Theo. XXVI.)

9. Prove that a perpendicular is the shortest line that can be drawn to a straight line from a point without it.



Let AB be a perpendicular to the line ES, and AC any oblique line. AB and AC with the segment BC will form a right-angled triangle.

Ang. ACB $<$ ang. ABC (Theo. X).

Why can not AB be equal to AC (Theo. XIII)? Why not greater (Theo. XVI)? If AB is neither equal to nor greater than AC, what follows?

Cor.—Prolong AB till BO is equal to AB. It is readily proved that CO $=$ AC. Hence AO, which is equal to twice AB, is less than AC $+$ CO, or twice AC.

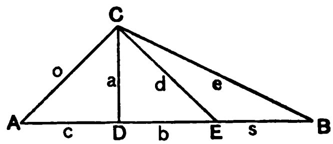
Therefore, a straight line is the shortest distance between two points.

Sch. 1.—This corollary is sometimes given as an axiom, and sometimes as the definition of a straight line.

Sch. 2.—Notice carefully the difference between this discussion and that proving Theo. VII.

10. Prove that if, in the last theorem, two oblique lines be drawn, the one cutting the given straight line the furthest from the foot of the perpendicular will be the longest. (Consult Theorems IX and XI).

NOTE.—It is thought best here to illustrate by examples the application of the equation to geometry. The problems will be understood readily even by those pupils who have not studied algebra.



11. The line CD is perpendicular to AB, and AD = DE.

To prove that CD is the shortest distance from the point C to the line AB, as has been already done in No. 9.

Using the letters attached to the several lines as general symbols for the *numerical* value of the lines, and applying Theo. XXVI,

(1) $d^2 = a^2 + b^2$ and

(2) $e^2 = a^2 + (b + s)^2$;

$\therefore a^2 < d^2$ and $a < d$

$a^2 < e^2$ and $a < e$.

As d and e are any oblique lines, we prove the perpendicular less than any other line.

(1) $d^2 = a^2 + b^2$ and

(2) $e^2 = a^2 + b^2 + 2bs + s^2$.

Sub. (1) from (2), $e^2 - d^2 = 2bs + s^2$;

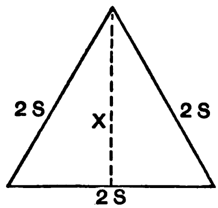
$\therefore e^2 > d^2$ and $e > d$.

We see that of any number of oblique lines, the long-

est is the one which cuts the given line furthest from the foot of the perpendicular.

Prove o = d.

12. To find the altitude of any equilateral triangle.



The known sides of the triangle we represent by 2S.

Let X = the altitude, and

$4S^2 - S^2 = X^2$;

$X^2 = 3S^2$;

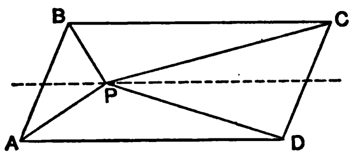
$X = S\sqrt{3}$.

Hence, the altitude is equal to the half side multiplied by the square root of 3.

13. What is the altitude of an equilateral triangle whose sides are each 12 feet?

14. What is the side of an equilateral triangle whose altitude is 12 feet? What is its area?

15. The two triangles, formed by drawing lines from a point within a parallelogram to the ends of two opposite sides, are, together, equal to one half the parallelogram.



SECTION VII.—POLYGONS.

DEFINITIONS.

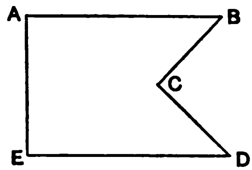
1. A polygon of five sides is a **pentagon;** one of six sides, a **hexagon;** of seven sides, a **heptagon;** of eight, an **octagon;** of ten, a **decagon;** of twelve, a **dodecagon.**

2. A **regular** polygon is both equilateral and equiangular. Squares and equilateral triangles are regular polygons.

3. The **perimeter** of a polygon is the sum of its sides.

4. A **convex polygon** is a polygon all of whose diagonals lie within the figure.

5. A **concave polygon** is a polygon which has at least one diagonal without the figure.

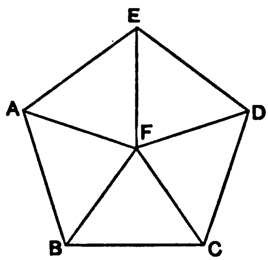


Thus, ABCDE is a concave polygon.

Ques.—What polygon can not be concave?

THEOREM XXVII.

The sum of all the interior angles of any polygon is equal to twice as many right angles, wanting four, as the figure has sides.



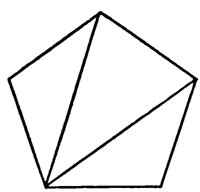
The point F is any point chosen within the polygon. To F, lines are drawn from all the vertices, forming the triangles EFD, DFC, etc.

How many of these triangles will be found in a polygon of n sides?

How many right angles in each (Theo. X)?

Give demonstration.

Use this figure also.

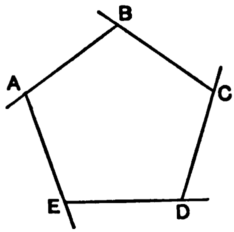


How many triangles when the polygon is thus divided?

If these figures are regular, and EF, DF, etc., bisect ang. E, ang. D, etc., how many degrees in each angle of the first? How many degrees in each angle of the second? What kind of triangle is each?

THEOREM XXVIII.

The sum of all the exterior angles of a convex polygon is equal to four right angles.

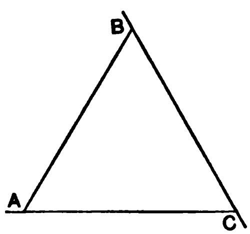


The polygon ABCDE has n sides, and therefore n vertices. At each vertex the sum of the interior angle and the exterior angle is two right angles (Theo. I). Hence, the sum of all the interior and exterior angles is $2n$ right angles. Of these (Theo. XXVII), all but four are interior. Hence, the sum of the exterior angles is four right angles.

Stated as an identical equation, $2n$ right angles $-$ $(2n - 4)$ right angles $=$ 4 right angles.

Sch.—By introducing the idea of motion, a very elegant demonstration of this theorem can be made and Theo. XXVII would be a corollary under it.

Apply a pencil to the line BA, with the point toward A. Move it along till its middle point is on A, then rotate from right to left till it coincides with AC. Move its center to C, and rotate till it coincides with BC. Move it along BC till its center is on B, and rotate till it coincides once more with BA. It has changed direction only as it measured the angles A, C, and B, and is now on BA, with its point toward A. It has, therefore, rotated 360°, while it measured the three exterior angles: therefore, the sum of the exterior angles is four right angles. The result would be the same if any other polygon were taken.

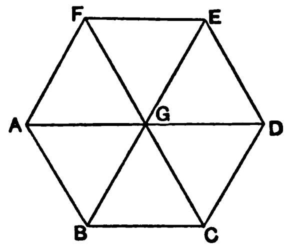


Cor. 1.—Hence, the sum of the interior angles of a triangle is two right angles; of a quadrilateral, four; of a pentagon, six; and so on.

Cor. 2.—By motion, the sum of the interior angles can be measured, as was that of the exterior.

THEOREM XXIX.

A regular polygon may be divided into as many equal triangles as it has sides.



Let ABCDEF be a regular polygon. Draw AG and BG, bisecting the angles A and B; also, draw GC, GD, etc., to the other angles.

Now, since the triangles AGB, BGC, have the side AB equal to BC by definition, and the side BG common, and the included angle ABG equal to the included angle CBG, the two triangles are equal throughout (Theo. XII). Hence, the angle GAB is equal to the angle GCB. But, by construction, GAB is half of BAF; therefore, GCB is half of the equal angle BCD, that is, BCD is bisected by GC. Hence, in the same manner, it may be shown that the triangles BGC and CGD are equal, and so of the other triangles in succession.

Therefore, a regular polygon, etc.

Def. 1.—The **center** of a regular polygon is the point of intersection of the lines which bisect its angles.

Def. 2.—The **apothegm** is the perpendicular from the center to any side of a regular polygon.

Cor. 1.—Prove that there is such a point, or center, equally distant from all the sides.

Cor. 2.—Since the area of each triangle is equal to half the product of its base by its altitude, the area of all the triangles taken together is equal to half the product of the sum of their bases by their common altitude; that is, the area of a regular polygon is equal to half the product of its perimeter by its apothegm.

EXERCISES.

1. To how many right angles are all the angles of a quadrilateral equal? Of a pentagon? Of a hexagon? How many degrees in one angle of a regular quadrilateral? Of a regular hexagon?

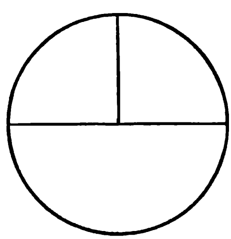
2. Show that three regular hexagons may be placed so as to have a common point and fill up the space around it. [360° ÷ 120° = 3.]

3. What other regular polygons have the same property?

4. How many diagonals can be drawn in a pentagon? How many in a hexagon? How many in a polygon of n sides? From each vertex there can be drawn $n - 2$ diagonals: how many vertices in the figure?

SECTION VIII.—OF THE CIRCLE.

DEFINITIONS.



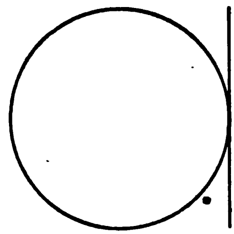
1. A **circle** is a plane figure bounded by a curve, called the **circumference,** all points in which are equally distant from a fixed point within, called the **center.**

2. A straight line drawn from the center to any point of the circumference is called the **radius.**

A double radius, that is, a

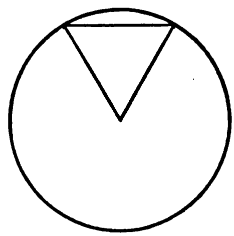
straight line passing through the center, and terminated both ways by the circumference, is called a **diameter.**

Cor.—In the same circle, all radii are equal; also, all diameters.



3. A **tangent** is a straight line which touches the circumference, but being produced does not cut it.

4. Any portion of the circumference is called an **arc.** The **chord** of an arc is the straight line joining its extremities. A chord produced one way beyond the circumference is called a **secant.**



5. A **segment** of a circle is the part contained by an arc and its chord.

A **sector** is the part contained by an arc and two radii drawn to its extremities.

6. Half a circle is called a **semicircle;** half a circumference, a **semi-circumference.** A quarter of a circle or of a circumference is called a **quadrant.**

7. An **inscribed angle** is one which has its vertex in the circumference, and is contained by two chords.

A **polygon** is said to be *inscribed* in a circle when all its sides are chords of that circle.

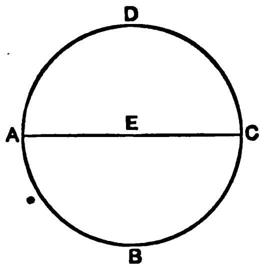
A **circle** is said to be *inscribed* in a polygon when all the sides of the polygon are tangents to the circle.

When one figure is inscribed in another, the latter is said to be *circumscribed* about the former.

Can a circle be circumscribed about every triangle? About every quadrilateral? Can two circles be drawn from the same center with the same radius?

THEOREM XXX.

A diameter divides the circle and its circumference into two equal parts.



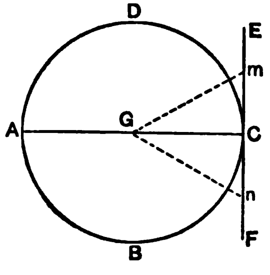
Let ABCD be a circle, of which E is the center and AC a diameter. It may be shown that the segment ABC is equal to the segment ADC, and the arc ABC to the arc ADC.

Let the segment ABC be applied to the segment ADC, the line AC remaining common; then the arc ABC will exactly coincide with the arc ADC; for if not, suppose some point in ABC to fall within or without ADC; then we shall have points in the circumference unequally distant from the center, which is contrary to the definition of a circle. Hence, the arc ABC is equivalent to the arc ADC, and the segment ABC to the segment ADC (Ax. 10).

Therefore, a diameter divides, etc.

THEOREM XXXI.

If a straight line is perpendicular to a diameter at its extremity, it is a tangent to the circle.



Let the straight line EF be perpendicular to the diameter AC at its extremity C. Then will EF be a tangent to the circle ABCD.

In EF, take any two points, *m* and *n*, one on each side of C, and connect them by straight lines with the cen-

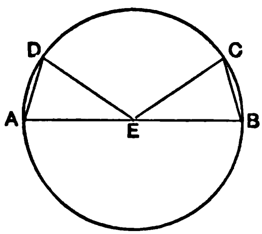
ter G. Now, since GCm is a right-angled triangle, the square of Gm is greater than the square of GC (Theo. XXVI); hence, Gm is greater than GC, that is, greater than the radius. Therefore, the point m, however near it may be to C, is necessarily without the circle. The same may be proved, in like manner, of the point n, or any other point in EF on either side of C. Hence, EF can meet the circumference only in the point C, and it is consequently a tangent (Def. 3, Sec. VIII).

Therefore, if a straight line, etc.

Cor.—A chord, drawn perpendicular to a tangent at the point of contact and terminating in the opposite arc, is a diameter.

THEOREM XXXII.

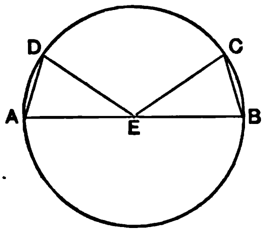
Equal arcs in a circle have equal chords, and subtend equal angles at the center.



Let AD and BC be equal arcs, and E the center of the circle. It is to be proved that the chords AD, BC are equal; also, the subtended angles AED, BEC.

Let the sector EBC be applied to the sector EAD, so that the line EB shall fall upon the line EA, the point E remaining common; then, since these lines are equal, being radii (Cor., Def. 2, Sec. VIII), the point B will fall upon the point A; and, since all points in the circumference are equally distant from the center, the arc BC will fall upon the arc AD; and these arcs being by hypothesis equal, the point C will fall upon the point D; therefore (Ax. 10), the chord BC will coincide with the chord AD, and be equal to it.

Cor.—Prove the same of equal arcs of equal circles.



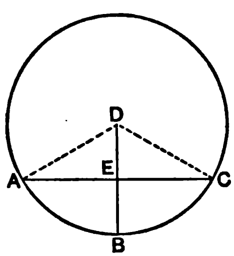
Again, in the triangles EAD, EBC, the two sides EA, ED, are equal to the two sides EB, EC, since they are all radii; and the base AD has just been proved equal to the base BC; therefore, the two triangles are mutually equiangular (Theo. XIX), and the angle AED is equal to the angle BEC (Cor., Theo. XII).

Hence, equal arcs in a circle, etc.

Sch.—If the whole circumference be divided into any number of equal arcs, and radii be drawn to all the points of division, the whole compass about the center will be divided into the same number of equal angles. Hence, *an arc may be taken as the measure of its subtended angle.* To this end the circumference is conceived to be divided into 360 equal parts, called *degrees*, each degree into 60 *minutes*, and each minute into 60 *seconds*. Then, whatever number of degrees, etc., an arc contains, the same number will denote the magnitude of the angle at the center which it subtends.

THEOREM XXXIII.

If a radius is perpendicular to a chord, it bisects the chord, and also the arc which the chord subtends.



Let the radius DB be perpendicular to the chord AC. It is to be proved that it bisects AC, and also the arc ABC.

Draw the radii DA, DC. Now, since the right-angled triangles AED, CED, have the hypotenuse AD equal to the hypotenuse CD,

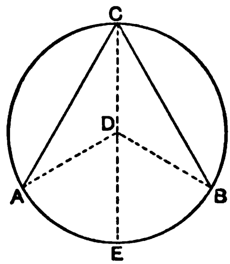
and the side ED common, it follows that the third side AE is equal to the third side CE (Cor. 3, Theo. XXVI); that is, AC is bisected in E.

Again, since the triangles AED, CED, are mutually equilateral, the angle ADE is equal to the angle CDE (Theo. XIX). Hence, the arc AB which measures the former angle is equal to the arc CB which measures the latter (Sch., Theo. XXXII).

Therefore, if a radius is perpendicular, etc.

THEOREM XXXIV.

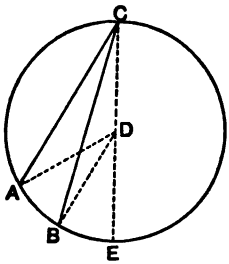
An inscribed angle is measured by half the arc on which it stands.



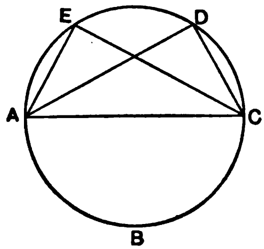
Let ACB be an angle inscribed in a circle. It is measured by half the arc AEB.

First, suppose the center D to be within the angle. Draw the diameter CE; also, join DA and DB. Now, since AD and DC are radii, the triangle ADC is isosceles; hence, the angles DAC, DCA, are equal (Theo. XIII). But the exterior angle ADE is equal to the sum of the two opposite interior angles DAC, DCA (Theo. X); it is consequently double of DCA; and since ADE is measured by the arc AE (Sch., Theo. XXXII), it follows that DCA is measured by half of AE. In the same manner it may be shown that DCB is measured by half of BE. Therefore, the whole angle ACB is measured by half of the whole arc AEB.

Next, let the center D be without the angle ACB. By the above demonstration, the angle ACE is meas-



ured by half the arc AE, and the angle BCE is measured by half the arc BE; therefore, BCA, which is the difference of these two angles, is measured by half the difference of the two arcs, that is, by half of AB.

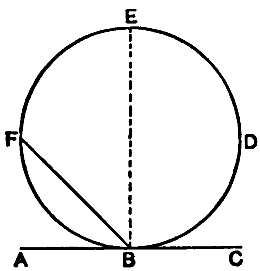


Cor. 1.—*Angles inscribed in the same segment,* as AEC and ADC, *are equal;* for they are measured by half the same arc ABC.

Cor. 2.—*Any angle inscribed in a semicircle is a right angle;* being measured by half a semi-circumference.

THEOREM XXXV.

An angle contained by a tangent and a chord is measured by half the intercepted arc.



Let AC be a tangent to a circle, and BF a chord drawn from the point of contact B. Then will the angle ABF be measured by half the arc BF, and the angle CBF by half the arc BDF.

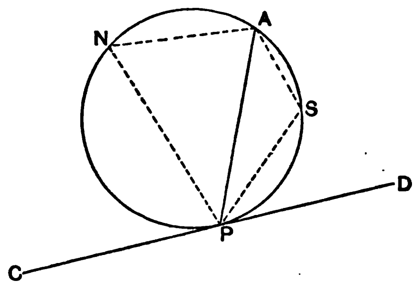
Draw BE perpendicular to AC, and it will be a diameter of the circle (Cor., Theo. XXXI). Now, since ABE is a right angle, it is measured by half the semi-circumference, BFE (Book I, Theo. V, Sch. 2); and since the part EBF is an inscribed angle, it is measured by half the arc FE (Theo. XXXIV);

therefore, the remaining angle ABF is measured by half the remaining arc BF.

Again, because CBE is measured by half the semi-circumference BDE, and EBF by half the arc EF, it follows that the whole angle CBF is measured by half the whole arc BDF.

THEOREM XXXVI.

Each angle contained by a tangent and a chord drawn from the point of contact is equal to the angle in the alternate segment.



How is APD measured?

How ANP?

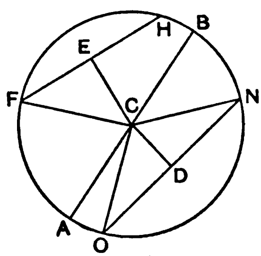
What is the inference?

Prove that APC is equal to ASP.

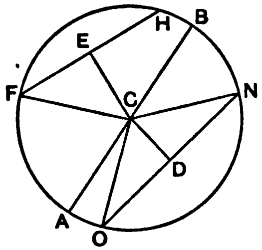
Cor.—Prove that NAS + NPS = 2 ∟'s, no matter where the points N and S be.

THEOREM XXXVII.

The diameter of a circle is longer than any other chord, and of two unequal chords the further from the center is the shorter.



In the circle C, let ON be any straight line not drawn through the center; then will any diameter AB be longer than it. Draw radii from the center C to O and N. Then, CO + CN = AB (Def. 2, Sec. VIII), and CO + CN > ON (?).



$\therefore$ AB $>$ ON.

FH and ON are chords, and FH is further from the center, therefore it is less than ON.

CE and CD are perpendiculars from the center to the two chords, and hence measure the distance of the chords.

But $\overline{CE}^2 + \overline{EF}^2 = \overline{CF}^2$ (?).

And $\overline{CD}^2 + \overline{DO}^2 = \overline{CO}^2$ (?).

Since CF and CO are radii, $\overline{CE}^2 + \overline{EF}^2 = \overline{CD}^2 + \overline{DO}^2$.

By hypothesis, CE $>$ CD.

$\therefore$ $\overline{CE}^2 > \overline{CD}^2$ and

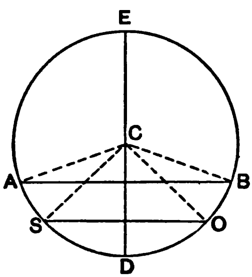
$\therefore$ $\overline{EF}^2 < \overline{DO}^2$ (?) and EF $<$ DO;

$\therefore$ 2EF $<$ 2DO;

$\therefore$ FH $<$ ON (Theo. XXXIII).

THEOREM XXXVIII.

Parallel secants, or chords, intersect equal arcs of the circumference.



In the circle C, AB and SO are parallel chords, therefore the arcs AS and BO are equal.

The arcs BD and AD are equal. Likewise OD and SD (Theo. XXXIII).

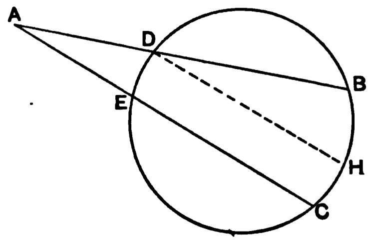
$\therefore$ arc AS $=$ arc OB (?).

Give demonstration also by rotating one side of the figure upon ED as an axis. Why would the point A fall some place on the half chord? Why not to the left of B? Nor the right?

Cor.—Through E and D draw parallels touching the circumference, and deduce a theorem.

THEOREM XXXIX.

The angle formed without a circle by two secants, is measured by half the difference of the intersected arcs.

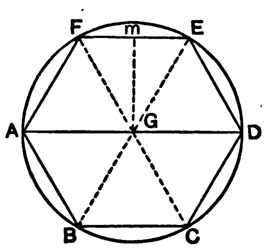


The secants AB and AC intersect at A, and the angle BAC is measured by half of BH, which is BC — DE.

DH is drawn parallel to AC, consequently the arc HC = the arc DE (Theo. XXXVIII), and BC — HC = BC — DE.

THEOREM XL.

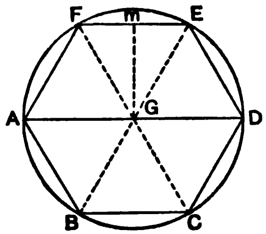
If a circumference be divided into any number of equal arcs, their chords will form a regular polygon.



Let the arcs AB, BC, etc., be equal each to each, then will the inscribed polygon be regular.

Draw the radii AG, BG, CG, etc. Comparing the triangles AGB, BGC, and CGD, and applying Theorems XIII and XXXII, we find AB = BC = CD, and ang. ABC = BCD. Similarly, all the sides and angles of the polygon are shown to be respectively equal, which answers the definition of a regular polygon. (Quote it.)

Cor. 1.—If the polygon ABCDEF be a hexagon, the ang. AGB, being measured by one third of a semicircumference, will be one third of two right angles



(Sch., Theo. XXXII). Hence, the other angles are each one third of two right angles. The triangle AGB is equilateral; that is:

The side of a regular hexagon is equal to the radius of the circumscribed circle.

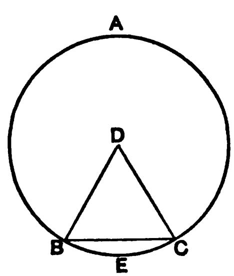
Cor. 2.—If from the center G, with the apothegm Gm as a radius, a circle be drawn, the side EF, and, consequently, all the sides of the regular polygon, will be tangents to that circle (Theo. XXXI), and the circle will be inscribed in the polygon (Def. 7, Sec. VIII).

Cor. 3.—The center of an inscribed or circumscribed regular polygon is the center of the circumscribed or inscribed circle.

Cor. 4.—The diagonals of a regular polygon drawn between vertices similarly situated with regard to each other, are equal each to each.

THEOREM XLI.

The area of a circle is equal to half the product of its radius by its circumference.



Let ABC be a circle of which DB or DC is radius.

If we conceive the whole circumference to be divided into equal arcs so small that we can not conceive of their length, the perimeter of the regular polygon formed by their chords (Theo. XL) will coincide with

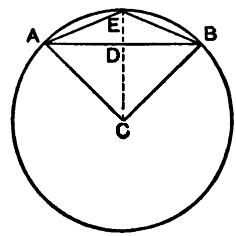
the circumference, and the area of the polygon will be equal to the area of the circle; also, the apothegm of the polygon will be equal to the radius of the circle. But the area of the polygon will be equal to half the product of its apothegm by its perimeter (Cor. 2, Theo. XXIX). Therefore, the area of the circle must be equal to half the product of its radius by its circumference.

Cor. 1.—In like manner, it may be shown that the area of any sector, as BECD, is equal to half the product of the radius into its arc.

Cor. 2.—The area of a segment BEC, less than a semicircle, may be found by subtracting the triangle BDC from the sector BECD. The area of a segment BAC, greater than a semicircle, may be found by adding the triangle BDC to the sector BACD.

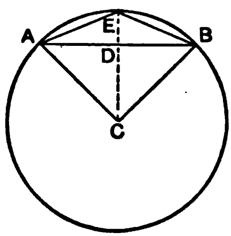
THEOREM XLII.

The radius and one side of a regular inscribed polygon being given, the side of a regular inscribed polygon of twice the number of sides may be found.



Let AB be the side of a regular inscribed polygon in a circle whose radius is CA or CB. Draw CE perpendicular to AB. This will bisect AB and also the arc AEB (Theo. XXXIII). Hence, the chords AE, EB (Theo. XL) will be sides of a regular inscribed polygon having twice as many sides as the first.

Now, if the lengths of CB and AB are given, we shall have, in the right-angled triangle CDB, the hypotenuse CB and the side BD; hence, we may find the other side DC (Cor. 2, Theo. XXVI). Subtracting this from



the radius CE, we shall have DE. Then, in the right-angled triangle BDE, we shall have the two sides BD, DE, from which we can find the hypotenuse EB (Cor. 1, Theo. XXVI), which is the side required.

Sch.—If the diameter be 1, the side of a regular inscribed hexagon will be $\frac{1}{2}$ (Cor. 1, Theo. XL). From this we can find the side, and consequently the perimeter, of a regular inscribed dodecagon; from that we can find the perimeter of a regular inscribed polygon of 24 sides, etc. By carrying this calculation on to polygons of an indefinitely large number of sides, it is found that the perimeter, though it increases at every step, never exceeds 3.14159, except by decimal figures beyond the 9; that is, beyond the fifth place of decimals. Hence, since the perimeter ultimately coincides with the circumference, it follows that the circumference can not differ from 3.14159, except by decimals beyond the fifth place. Disregarding these, we may conclude that the circumference of a circle, whose diameter is 1, is 3.14159.

Cor.—If the diameter is 1, the area is equal to $\frac{\frac{1}{2} \times 3.14159}{2}$ (Theo. XLI), which by reduction is .7854.

Sch.—There are three modes of angular measurement.

The first refers all angles to an arbitrary unit, 1°.

The second refers all angles to the *right angle* as the unit—one of the rare instances in which we have a *natural* unit. [Postpone till review.]

The third mode has for its unit the *radian;* that is, the arc which is equal in length to the radius. It is obtained thus:

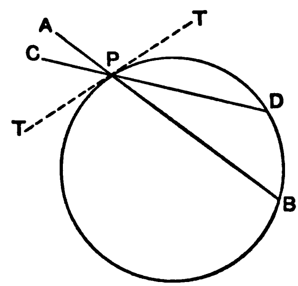
$$2\ R\pi = 360^\circ;$$
$$\therefore R\pi = 180^\circ;\ \therefore R = 180^\circ \div \pi;$$
$$R = 57^\circ +,\ \text{or radian.}$$
$$180^\circ = \text{radian} \times \pi;$$
$$1^\circ = \frac{\pi}{180} \times \text{radian.}$$

EXERCISES.

1. A chord of a circle is 60 feet, and its distance from the center 20 feet: what is the area of the circle?

2. The hypotenuse of a right-angled triangle inscribed in a circle is 50 feet. Find the area of the circle.

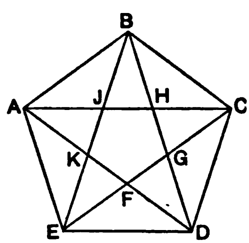
3. Two secants, AB and CD, intersect at P, a point on the circumference. Prove that the angle APD is measured by half the sum of the arcs PD and PB. Draw tangent TPT, and apply Theorem XXXV. How is the angle measured which is formed by lines intersecting within a circle? On the arc? Without the arc?



4. The diagonals of a regular pentagon will inclose a regular pentagon.

The diagonals are all equal; and the triangles AED, ABE, ABC, BCD, CDE are isosceles.

The angles F, G, H, J, K are equal (?). Complete.



5. Examine the figure enclosed by the diagonals of a regular hexagon, drawing no diagonal through the center, and deduce and prove a theorem.

SECTION IX.—PROBLEMS IN CONSTRUCTION.

To this point, when we desired to use, for illustration or demonstration, a line or figure drawn in a certain way, we drew a line or figure which we assumed as the thing wanted.

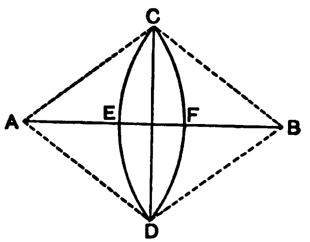
We are now to learn how to draw lines under certain geometric conditions, having the ruler and compasses as instruments with which to work. For example, we have dealt with lines as perpendicular to other lines, or as parallel to one another. We must now learn to draw them so. We have assumed a point as the center of a given line. We must now find the center.

This branch of Geometry is, therefore, the art of constructing figures and determining points, under given conditions, and according to the principles of the science of Geometry, as already laid down.

It teaches the solution of problems whose *answers* are points, or lines, or figures.

PROBLEM I.

To bisect a given straight line.



SOLUTION.—Let AB be the given line. From A as a center, with a radius obviously greater than one half of AB, describe the arc CFD. From B as a center, with the same radius, describe the arc CED. Join the points of intersection C and D.

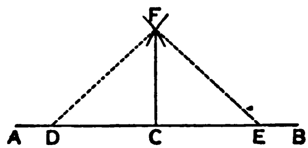
Since the opposite sides of the quadrilateral ACBD are equal, it is a parallelogram (Theo. XXI). But the

diagonals of a parallelogram bisect each other (Theo. XXII). Therefore, CD bisects AB.

Cor.—If a line has two points each equally distant from the extremities of another line, the first line is perpendicular to the second at its center.

PROBLEM II.

At a given point in a straight line, to erect a perpendicular to that line.

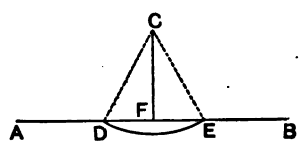


Solution.—Let AB be an indefinite straight line, and C the given point in it. Take the points D and E equally distant from C. From D as a center, with a radius greater than DC, describe an arc; and from E as a center, with the same radius, describe another arc intersecting the former at some point F. Now, draw FC, and it will be the perpendicular required.

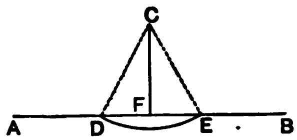
Join FD and FE. Then, since the triangles DFC, EFC, are by construction mutually equilateral, the angles FCD, FCE, are equal (Theo. XIX); they are, therefore, right angles, and CF is a perpendicular to AB at the point C (Defs. 2 and 3, Sec. IV).

PROBLEM III.

To draw a perpendicular to a straight line from a given point without.



Solution.—Let C be the given point without the straight line, AB. From C as a center, with a radius greater than the shortest

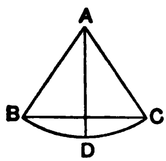


distance to the line, draw an arc cutting AB in D and E. Bisect DE in F (Prob. I). Then draw CF, and it will be the perpendicular required.

Join CD and CE. Now, since the triangles DCF, ECF, are mutually equilateral, the angles CFD, CFE, are equal (Theo. XIX); they are, therefore, right angles, and CF is a perpendicular to AB from the point C.

PROBLEM IV.

To bisect a given angle.

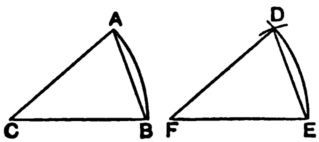


Solution.—Let BAC be the given angle. From A as a center, with any radius, draw an arc, BDC, intersecting both sides of the angle. Then draw AD perpendicular to the chord BC (Prob. III), and it will bisect the angle.

For, since the perpendicular AD is radius, it bisects the arc BDC (Theo. XXXIII). Therefore, the arcs BD, DC, being equal, the subtended angles BAD, DAC, must also be equal (Theo. XXXII), and the angle BAC is bisected by AD.

PROBLEM V.

At a given point in a straight line, to make an angle equal to a given angle.

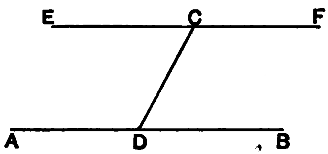


Solution.—Let ACB be the given angle, and F the given point in a straight line FE. From C as a center, with any radius as BC, describe an arc BA

intersecting both the sides of the angle; also draw the chord BA. Then, from F as a center, with a radius FE equal to CB, draw an indefinite arc; and from the center E, with a radius equal to the chord BA, draw an arc intersecting the other at D; also join DE and DF. The angle DFE is the angle required. For, by equality of the triangles DEF, ABC, the angle F is equal to the angle C (Theo. XIX).

PROBLEM VI.

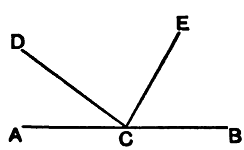
Through a given point, to draw a straight line parallel to a given line.



SOLUTION.—Let AB be the given straight line, and C the given point. In AB take any point D, and join CD. Through C draw EF, making the angle ECD equal to the angle CDB (Prob. V). These being alternate angles, the straight line EF must be parallel to AB (Cor. 1, Theo. III).

PROBLEM VII.

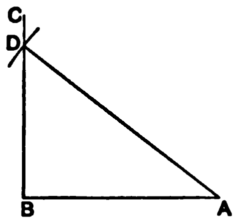
Two angles of a triangle being given to find the third.



SOLUTION.—At any point C, in a straight line AB, make ACD equal to one of the given angles (Prob. V), and DCE equal to the other; then will ECB be equal to the third angle. For the sum of the three angles at C is two right angles (Cor. 1, Theo. I); which is also the sum of the three angles of a triangle (Theo. X).

PROBLEM VIII.

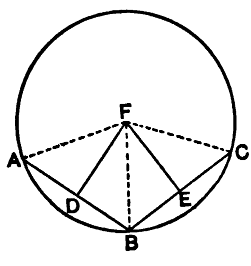
Given the hypotenuse and one side of a right-angled triangle, to construct the triangle.



SOLUTION.—Draw AB equal to the given side. At B erect a perpendicular BC (Prob. II). From A as a center, with a radius equal to the given hypotenuse, describe an arc intersecting BC at D. Join AD. It is evident that ABD is the triangle required.

PROBLEM IX.

To draw a circle through three given points.



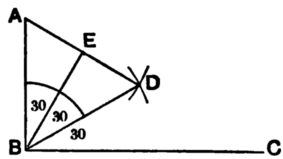
SOLUTION.—Let A, B, and C be the given points. Draw the straight lines AB and BC, and from their middle points erect the perpendiculars DF and EF. Also, join AF, BF, and CF.

Now, because the triangles AFD, BFD, have the side AD equal to the side BD, and the side DF common, and the included angle ADF equal to the included angle BDF, the third side AF is equal to the third side BF (Theo. XII). In the same manner it may be shown that BF is equal to CF. Therefore, the three lines AF, BF, and CF are equal; and if, from F as a center, with one of these equals as radius, a circle be drawn, it will pass through the three points A, B, and C.

Cor.—Hence, the center of a circle may be found by erecting perpendiculars at the middle points of any two chords not parallel. Their intersection is the center of the circle.

PROBLEM X.

To trisect a right angle.



SOLUTION.—Let ABC be a right angle. From A as a center, with AB as a radius, draw an arc, and from B as a center, with the same radius, draw an arc intersecting the first at D, and draw BD. Then bisect the angle ABD by BE. The angles ABE, EBD, and DBC are equal each to each(?); therefore, the right angle at B is trisected.

Ques.—What kind of a triangle is ABD, and how many degrees in each of its angles?

PROBLEM XI.

Having the three sides of a triangle given, to construct the triangle.

Ques.—How many different triangles can be formed by different arrangements of the sides?

Ques.—In what cases will this problem be impossible?

PROBLEM XII.

Having the four sides of a quadrilateral, to construct the figure.

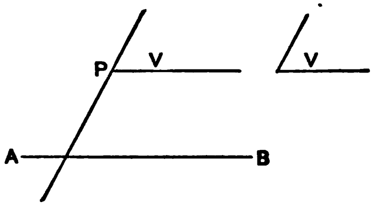
Ques.—Can more than one be formed?

PROBLEM XIII.

Given two sides of a parallelogram and the included angle, to construct the parallelogram.

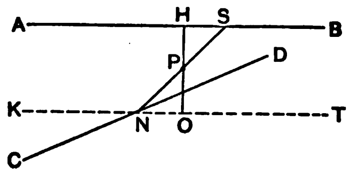
PROBLEM XIV.

From a given point without a straight line, to draw a line making an angle with the line equal to a given angle.



PROBLEM XV.

Through a given point between two straight lines not parallel, to draw a straight line which shall be bisected at that point.



Through P, the given point, draw a perpendicular to AB, one of the given lines, and make PO = PH, and draw KT parallel to AB. Every line from AB to KT through P is bisected at P. From N, where KT intersects CD, draw NS through P, and NP = PS; ∴ NS is the required line.

Ques.—Will KT necessarily intersect CD?

PROBLEM XVI.

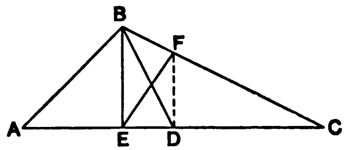
Construct a square which is double the area of a given square. (Theo. XXVI.)

PROBLEM XVII.

Construct a square which is equal to the sum of two, three, or any given number of squares. (Theo. XXVI.)

PROBLEM XVIII.

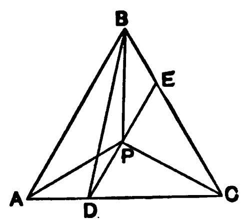
To divide a triangle into two equivalent parts by a line drawn from any point in any side.



Let F be the point given. Draw BD to the center of the base. It will divide the triangle ABC into two equivalent triangles. Draw FD, and BE parallel to FD. Draw BD. The triangles EDF and BDF are equivalent (?); $\therefore$ FDC + EDF = FDC + BDF. Complete.

PROBLEM XIX.

To divide a triangle into three equivalent parts by lines drawn from the vertices of the angles to the same point within the triangle.



Lay off AD $= \frac{AC}{3}$. Draw DE parallel to AB, and P, the center of DE, will be the required point.

The triangle ABD = one third of ABC (?).

Triangle ABD = triangle ABP (?).
Triangle APD = triangle BPE (?).
Triangle DPC = triangle EPC (?). Complete.

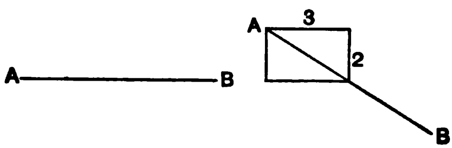
PROBLEM XX.

Divide a given line into two such parts that the square upon one of them will be half the square upon the whole line.

Construct a square, and divide the given line in the ratio of the side and diagonal.

PROBLEM XXI.

Given the diagonal of a rectangle and the ratio of its adjacent sides, to construct the rectangle.



Suppose the adjacent sides to have the ratio of 2 to 3. Construct a rectangle with its adjacent sides respectively twice and three times some assumed line. Upon the diagonal of this rectangle, produced if need be, lay off the given diagonal AB, and complete the rectangle.

PROBLEM XXII.

To draw a circle equivalent to the sum of two given circles.

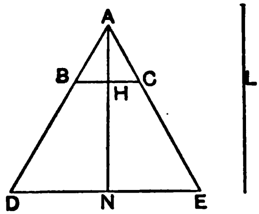
Notice the relation between a circle whose diameter is the hypotenuse of a triangle, and the sum of the circles whose diameters are the other two sides. The ratio of a square to its inscribed circle is constant.

PROBLEM XXIII.

To draw a tangent to an arc at a given point of the arc. (Theo. XXXI.)

PROBLEM XXIV.

Given the altitude of an equilateral triangle, to construct the triangle.



Construct an equilateral triangle ABC, and prolong its altitude, AH, till AN equals L, the given altitude. Through N draw a line parallel to BC, and prolong AB and AC to D and E. ADE is the triangle required (?).

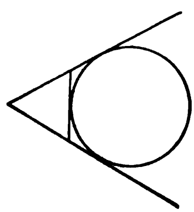
PROBLEM XXV.

To inscribe a circle in a given triangle. (Sec. V, Exercise 7.)

Ques.—When will the centers of the inscribed and circumscribed circles coincide? When will the center of the circumscribed circle be without the triangle?

PROBLEM XXVI.

Produce two sides of a triangle, and draw a circle to which the two sides produced, and the third side, will be tangents.



Def.—A circle thus drawn is said to be *escribed.*

Ques.—How many escribed circles may be drawn to a given triangle?

SECTION X.—LOCI.

1. As in other sciences, so in Geometry, it is common for a word to be chosen as the sign of some general idea. In this technical use of the word all other meanings are excluded, and it stands for that alone for which it is allowed to stand. These words are commonly taken from the Greek or Latin tongue.

2. **Locus,** with its plural **Loci,** is such a word; and we will illustrate by examples the general idea of which it is the sign.

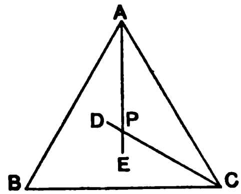
3. If it be required to find a point at a given dis-distance, R, from another point, C, an unlimited number of solutions can be given, for all the points in one plane, answering the single condition, are found in the circumference of a circle whose center is C and whose radius is R, and every point in such curve fulfills this condition.

4. If it be required to locate a point in a plane at a distance, D, from a given straight line, AB, it is obvious that a pair of parallels, each drawn at a distance, D, from the given line, will contain all such points, and that every point in either of the parallels fulfills the condition.

5. If it be required to find a point equally distant from the extremities of a given line, a brief investigation (Theo. IX) has shown that the line bisecting the given line at right angles contains all such points, and that every point in it is thus equally distant.

6. Any line, or series of lines, which may be regarded as made up of all the points in a plane which have some common property, is called a **plane locus.**

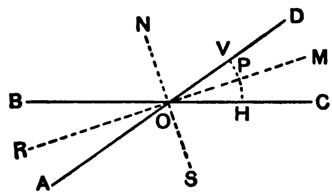
7. When two loci intersect, the common point possesses the properties of both.



Thus, if the line DC bisect the angle BCA, it is the locus of all the points within equally distant from BC and AC (Sec. V, Ex. 7). If AE bisect BAC, it is the locus of all the points within equally distant from BA and AC (Theo. XVIII). Hence, the point P, where these loci intersect, has the properties of both, and is equally distant from BC, AC, and BA.

THEOREM XLIII.

The locus of a point equidistant from two intersecting lines is the pair of lines, at right angles to one another, which bisect the angles made by the given lines.



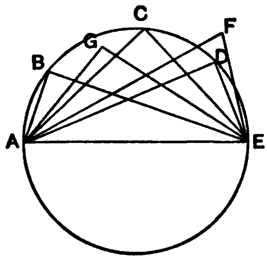
The angle DOC is bisected by the line RM. Any point, as P, is equally distant from DA and BC (Theo. XVIII).

Similarly, any point in the bisecter NS is proved equally distant from DA and BC. NS and RM intersect at right angles (?).

NS and RM are, therefore, the required locus (Section X, 6).

THEOREM XLIV.

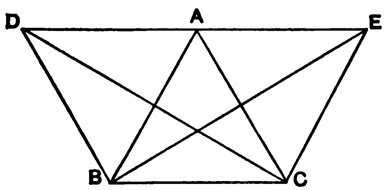
The locus of the vertex of the right angle, in a triangle having a given hypotenuse, is a circumference whose diameter is the given hypotenuse.



If the vertex of the triangle be at B, C, D, or at any other point in the arc, of what size is the angle? If it be within the arc, as at G? If it be without, as at F? If the vertex can be neither within nor without the arc, where is it?

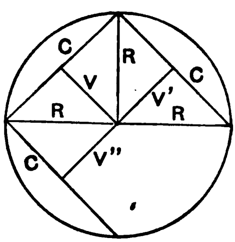
THEOREM XLV.

A triangle on a given base has a given area, what is the locus of its vertex?



Let BC be the given base, and the area of the triangle BAC be the given area. How must the line DE, containing the vertices D, A, and E, be drawn if the triangles BDC, BAC, and BEC are equal in area?

THEOREM XLVI.



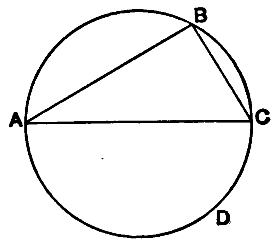
In a circle whose radius is R, *the locus of the centers of all the equal chords,* 2C, *is a circle having the same center, and whose radius is* $\sqrt{R^2 - C^2}$.

$$V = \sqrt{R^2 - C^2} = V' = V''.$$

Suppose that C becomes zero, what is the value of V? Suppose C = R? Can C be greater than R? Explain results from the figure and from the formula.

THEOREM XLVII.

The locus of all the points the sums of the squares of whose distances from two fixed points is constant, is a circumference.



In the circumference ABCD, wherever B be taken, $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$ (Theo. XXVI and Theo. XXXIV, Cor. 2).

Explain the effect upon the equation when the moving point B is at C; at A; midway between.

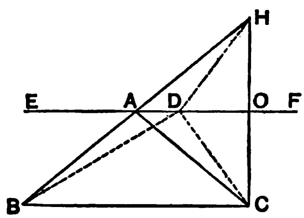
THEOREM XLVIII.

The locus of the centers of all the circles which can pass through two given points is a straight line.

Locate it.

THEOREM XLIX.

Of all equivalent triangles having a given base the isosceles has the least perimeter.

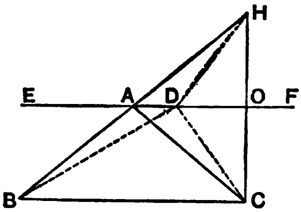


Let BAC be an isosceles triangle, and BDC an equivalent triangle with the same base, BC, but not isosceles.

To prove BA + AC < BD + DC.

As the triangles have equal areas, the vertex D is found in the line EAF, drawn parallel to BC (Theo. XXIV, Cor. 2).

From C, draw a perpendicular to EF, and prolong



it till OH = OC. Then, every point in EF will be equally distant from H and C (Theo. IX).

Hence, AH = AC and DH = DC;

$\therefore$ BA + AC = BA + AH,

And BD + DC = BD + DH.

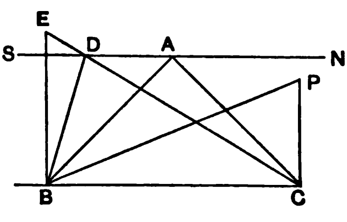
But BA + AH is a straight line. For, ang. HAO = OAC, by equality of triangles; OAC = ACB, by parallelism of lines; ACB = ABC (Theo. XIII), and ABC = BAE. These angles, HAO and BAE, being equal, and EF being a straight line by construction, BH must be a straight line.

Therefore, BA + AH, or BH, is shorter than any broken line joining the same points; and BA + AC is shorter than the sum of any other pair of lines, such as BD and DC.

Cor.—There is but one straight line from B to H, and hence but one shortest broken line from B to C, touching EF.

THEOREM L.

Of all triangles having equal perimeters and equal bases, the isosceles has the greatest area.



Let BAC be an isosceles triangle, and BPC a triangle not isosceles, equal in perimeter to BAC, and having the same base. Then will the area of BAC be greater than that of BPC.

It should be noticed that BPC is any triangle, not isosceles, with a base BC, and equal in perimeter to BAC, and its vertex *above* BC.

As these two have a common base, it is sufficient to prove that BAC has the greater altitude.

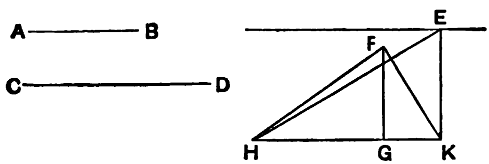
Ques.—Can the vertex P be on the parallel SN? (See Theo. XLIX.)

Ques.—Can the vertex P be above SN? (Theorems XI and XLIX.)

Ques.—If the vertex P is neither on SN, nor above it, where is it, and what conclusion?

THEOREM LI.

Of all triangles having two sides respectively equal to two given lines, the greatest area is when these two sides form a right angle.



AB and CD are the two given lines.
CD = HK.
AB = EK = FK.
EK is perpendicular to HK, and FK is oblique.
FG $<$ FK (Theo. XXVI).
$\therefore$ FG $<$ EK.
$\therefore$ HK $\times$ FG $<$ HK $\times$ EK.

Ques.—What angle must FK make with HK so that the triangle HFK may contain half the area of HEK?

EXERCISES.

1. Given the three angles of a triangle, to construct the triangle.

Ques.—When is it impossible?

2. Given two angles and the included side, to construct the triangle.

Ques. When is this impossible?

3. Given two adjacent sides and the adjacent diagonal of a parallelogram, to construct the parallelogram.

4. To draw a diameter of a given circle.

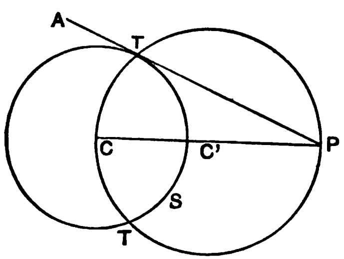
5. To inscribe a square in a given circle.

6. To inscribe a regular hexagon. (Cor. 1, Theo. XL.)

7. To inscribe an equilateral triangle.

8. Through a given point in a circle, to draw a chord which shall be bisected at that point. (Theo. XXXIII.)

9. At a given point without a circle, to draw a tangent to the circle.



From C, the center of the circle AS, draw a line to P, the given point; and with C′, the middle point of CP, as a center, and CC′ as a radius, describe a circle. It will cut the given circle in two points, T and T. Complete.

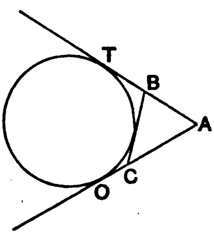
10. At any point in the circumference of a circle circumscribing a square, show that one of the sides subtends an angle three times as great as any other.

11. Two tangents to a circle from a point without are equal. (Theorems XXXV and XV.)

12. If a quadrilateral is circumscribed about a circle,

the sum of two opposite sides is equal to the sum of the other two.

13. If AT, AO, and BC are tangents, prove that the perimeter of ABC is 2AT, however BC may be drawn.



Sch.—Write this out in two ways as a general theorem.

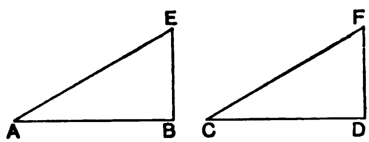
14. Each exterior angle of a quadrilateral inscribed in a circle is equal to the interior angle whose vertex is opposite to its own.

15. At the four corners of a square acre lot are four trees. How may the area of the lot be doubled, and still remain a square, without including the trees?

16. With a given radius, to describe a circle tangent to two given tangent circles.

17. Two right-angled triangles which have any two corresponding sides equal, are equal.

Let the equal sides be the hypotenuse and one leg.

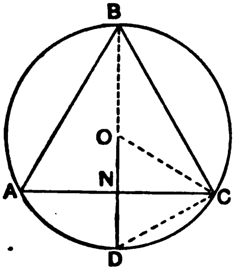


Place AB upon its equal CD. EB will fall upon FD, because of the right angles. If E does not fall upon F, there will be two equal oblique lines on the same side of the perpendicular. Can there be?

18. If the diameter of a circle be one of the equal sides of an isosceles triangle, the base will be bisected by the circumference.

19. From any point within a quadrilateral, not the intersection of the diagonals, the sum of the distances to the four angles is greater than the sum of the diagonals.

20. How does the side of an inscribed equilateral triangle divide a radius drawn through its center?



Prove ang. CDB = COD, each 60°. Prove OCA = ACD, and each 30°.

Then the triangle OCD is equilateral, and AC bisects the angle OCD. What follows?

21. What is the ratio of the area of a circumscribed equilateral triangle to that of an equilateral triangle inscribed in the same circle?

The area of the former is $2R\sqrt{3}$, or base, $\times \frac{3R}{2}$, or half altitude $= 3R^2\sqrt{3}$.

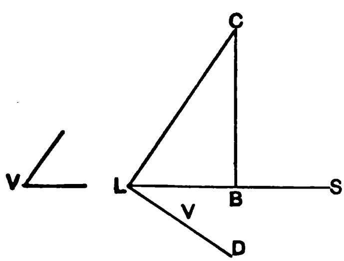
The area of the latter is $R\sqrt{3}$, or base, $\times \frac{3R}{4}$, or half altitude $= \frac{3R^2}{4}\sqrt{3}$. Ratio is 4.

In determining the areas, see Ex. 5, Sec. V, and Ex. 20, X.

22. To describe a circle about a given triangle.

23. To describe a circle about a given rectangle.

24. On a given line, to describe a segment of a circle which shall contain a given angle.

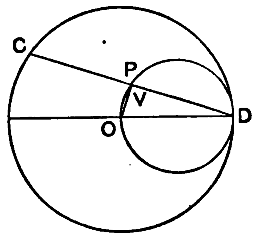


The given line is LS, and the given angle is V. Construct the angle SLD equal to V. Erect a perpendicular to LD at L, and bisect LS with the perpendicular BC. C, the point of intersection, will be the center of the circle.

Draw the circle and complete the solution by showing why it contains the required segment.

25. If a circle is described on the radius of another circle, any line from the point of contact to the outer arc is bisected by the inner arc.

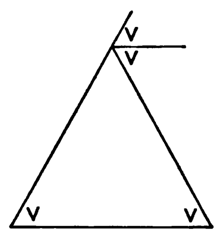
CD is any line so drawn. How is the angle OPD measured? P is the center of CD (Theo. XXXIII).



What is the **locus** of the centers of all the chords which can be drawn from D to points in the circumference?

26. The angle formed by the intersection of two chords is measured by half the sum of the arcs included by its sides.

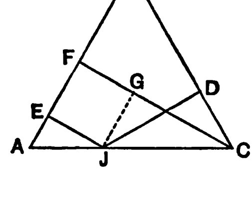
27. A straight line bisecting the angle exterior to the vertical angle of an isosceles triangle, is parallel to the base.



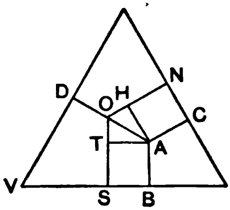
28. If BC is the base of an isosceles triangle ABC, and BD is drawn perpendicular to AC, the angle DBC is equal to one half the angle A.

29. Divide a circle into two parts, such that the angle contained in one segment shall equal twice the angle contained in the other segment. Apply Ex. 24.

30. If from any point in the base of an isosceles triangle perpendiculars are drawn to the sides, their sum is constantly equal to the perpendicular from either angle at the base to the opposite side. The point is J. JG is perpendicular to FC. Prove JD = GC. Complete.



31. If from any point within an equilateral triangle perpendiculars are drawn to the three sides, their sum is constant, and equal to either altitude.

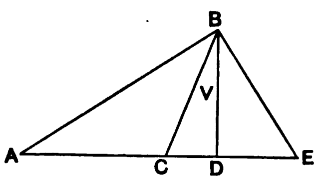


The ang. TOA = ang. V = 60°. ATO is a right angle; therefore, TAO = 30°, ∴ TO = $\frac{AO}{2}$ (Ex. 5, Sec. V).

Similarly, OH = $\frac{AO}{2}$.

Hence, in moving the point from A to O, AC increases OH; AB increases OT, and AD decreases AO. The sum is, therefore, the same as before. Prove it equal to the altitude.

32. In any triangle, if a line be drawn bisecting the vertical angle, and also a perpendicular be drawn from this angle to the base, these two lines will form an angle equal to half of the difference of the angles at the base.



(1) $BDE = A + \frac{B}{2} + V$ (?).

(2) $BDC = E + \frac{B}{2} - V$ (?).

(2) — (1) = (3), $E - A - 2V = 0$.

(4), $V = \frac{E - A}{2}$.

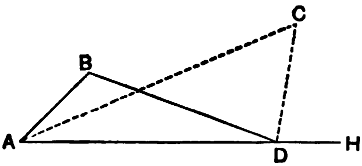
33. If an exterior angle of a triangle is bisected, and also one of the interior and remote angles, the angle contained by the bisecting lines is equal to half the other interior and remote angle of the triangle.

The angles BAD and BDH are bisected.

I. The bisectors will meet at some point; for, if they will not, they are parallel. But, if these lines are parallel, the sum of CAD + CDA = 2L's. On the

contrary, it is less. For, BDH > BAD. Therefore, CDH > CAD.

$\therefore$ CDH + CDA > CAD + CDA.



Complete this part.

II. (1) $B + \frac{BAD}{2} = C + \frac{BDH}{2}$ (?);

(2) $\frac{BDH}{2} = C + \frac{BAD}{2}$ (?);

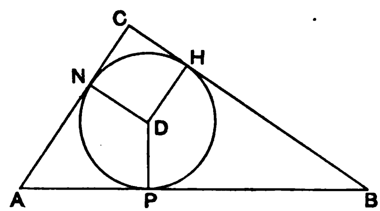
(3) $B + \frac{BAD}{2} = C + C + \frac{BAD}{2}$;

(4) $B = 2C$; (5) $C = \frac{B}{2}$.

34. If a circle is inscribed in a right-angled triangle, the sum of the two shorter sides minus the hypotenuse is equal to the diameter of the circle.

Apply Ex. 11, and prove BH + AN = AB.

CN and CH are each equal to radius (?).

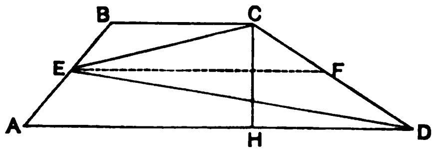


35. If from any point in the base of an isosceles triangle parallels to the sides are drawn, a parallelogram is formed whose perimeter is equal to the sum of the equal sides of the triangle.

36. Every line passing through the intersection of the diagonals of a parallelogram bisects the parallelogram.

37. If one leg of a right-angled triangle is double the other, the perpendicular from the vertex upon the hypotenuse divides it into parts which are to each other as 4 and 1. (See Theo. XXVI.)

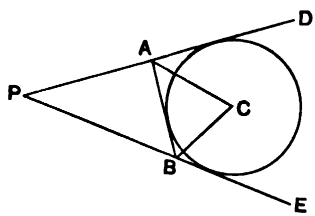
38. Connect the ends of one of the non-parallel sides of a trapezoid with the center of the other one, and prove the triangle thus formed equal to half the trapezoid.



BC and AD are the parallel sides of the trapezoid ABCD. Draw EF joining the centers of the non-parallel sides; also draw EC and ED, and the triangle CED is equal to one half the trapezoid. From C let fall CH, the altitude of the trapezoid.

Prove $EF = \frac{BC + AD}{2}$.

Then, the trapezoid is equal to EF multiplied by CH, the distance between the parallels, and the sum of the altitudes of the triangles EFC and EFD. Complete.



39. Prove that however the third tangent AB be drawn, the angle ACB is constant.

P is constant.

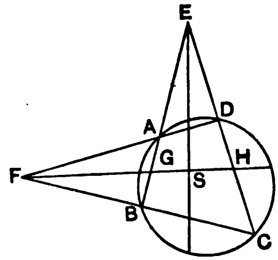
∴ PAB + PBA is constant.

∴ DAB + EBA is constant. Complete.

40. When the distance between the centers of two circles is less than the sum of their radii, how many tangents can be drawn common to the circles? When such distance is equal to the sum of the radii, how many? When greater, how many?

41. The bisectors of the angles contained by the opposite sides produced of an inscribed quadrilateral intersect at right angles.

Ang. EAD = ang. FAB = ang. BCD (?) [Sec. VIII, Ex. 3].



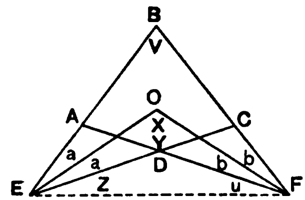
Ang. EGH = ang. FAB + ang. AFG (?).

Ang. EHG = ang. BCD + ang. HFC (?).

∴ Ang. EGH = ang. EHG.

∴ the triangle EGH is isosceles. Complete.

42. ABCD is a quadrilateral. The opposite sides produced, meet as in the previous theorem. Prove that the angle formed at O, by the bisectors EO and FO, is half the sum of ABC and ADC.



Draw EF.

For brevity, use algebraic symbols.

1. $Y = 2L\text{'s} - Z - U$;
2. $X = 2L\text{'s} - A - B - Z - U$:
3. $V = 2L\text{'s} - 2A - 2B - Z - U$;

$$Y + V = 4L\text{'s} - 2A - 2B - 2Z - 2U;$$

$$\frac{Y + V}{2} = 2L\text{'s} - A - B - Z - U;$$

$$\therefore \text{ see (2) } X = \frac{Y + V}{2}.$$

Cor.—If the given quadrilateral is capable of being inscribed within a circle, the angle at O is a right angle (?).

Ques.—What relation does the previous theorem bear to this one?

43. The locus of a point on one side of a given straight line, at which that line subtends a constant angle, is an arc of which that line is the chord. (Theo. XXXIV, Cor. 1, Book I.)

BOOK II.

PROPORTIONS OF MAGNITUDES LYING IN THE SAME PLANE.

SECTION XI.—PROPORTION.

Geometrical magnitudes are brought into the realm of common speech and thought by giving them numerical representation; as when we state that the line AB is ten inches long, and that the triangle ACB contains fifty square inches. But, in order to understand these assertions, one must have in his storehouse of ideas a mental inch and square with which to measure the quantity.

In the use of such expressions as 10 miles, 7 degrees, 12 acres, 6 years, we think and speak in ratios (Def. 12, Sec. II).

In Geometry, as in Algebra and Arithmetic, many important results are obtained by placing two ratios which are equal in the style of an equation or proportion.

Transforming this equation, we obtain results different from its original reading. These results must be true of the figure or figures, whose lines afforded the equal ratios. Hence, the algebraic expression must have a geometric interpretation.

The terms ratio and proportion have already been defined (Sec. II, Defs. 12 and 13).

The first and third terms of a proportion—*i. e.*, the two numerators—are called **antecedents**; the second and fourth, **consequents.** The first and fourth are called **extremes**; the second and third, **means.**

THEOREM I.

In any proportion, the product of the extremes is equal to the product of the means.

Let the proportion be $a:b = c:d$; or $\frac{a}{b} = \frac{c}{d}$;

$\therefore$ $ad = cb$. But a and d are extremes; b and c, means.

THEOREM II.

In any proportion, the first term is to the third as the second is to the fourth.

$a:b = c:d$; then, $\frac{a}{b} = \frac{c}{d}$. Multiply by $\frac{b}{c}$, and

$$\frac{ab}{bc} = \frac{bc}{cd}, \text{ or } \frac{a}{c} = \frac{b}{d}, \text{ or } a:c = b:d.$$

Def.—This mode of reading a proportion is called **alternation.**

THEOREM III.

If $ad = bc$, then $a:b = c:d$, or $a:c = b:d$.

Divide both terms of the given equation by *bd*, and then by *cd*, and the results are $\frac{a}{b} = \frac{c}{d}$ and $\frac{a}{c} = \frac{b}{d}$, or $a:b = c:d$ and $a:c = b:d$.

(Write out the theorem.)

THEOREM IV.

If $a:b = c:d$ and $e:f = g:h$, then $a + c + e + g : b + d + f + h = a:b$.

$$\begin{aligned} ad &= bc. \\ af &= be. \\ ah &= bg. \\ ab &= ab. \end{aligned}$$

Adding, $a(b + d + f + h) = b(a + c + e + g)$.

Then (Theo. III), $(a + c + e + g) : b + d + f + h = a : b$; that is, in a set of continued proportionals, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

Cor.—If any number of fractions are equal each to each, the sum of the numerators divided by the sum of the denominators is equal to any one of the fractions As in this theorem, $\frac{a + c + e + g}{b + d + f + h} = \frac{a}{b}$, or $\frac{c}{d}$.

THEOREM V.

$$
\begin{array}{ll}
\text{If} & a : b = c : d \\
 & ad = bc \\
\text{Adding,} & \underline{ac = ac} \\
 & a(c + d) = c(a + b); \\
\therefore (1) & a : c = a + b : c + d. \\
 & ad = bc \\
\text{Adding,} & \underline{ab = ab} \\
 & a(b + d) = b(a + c); \\
\therefore (2) & a : b = (a + c) : (b + d). \\
 & ad = bc \\
\text{Subtracting,} & \underline{ab = ab} \\
 & a(d - b) = b(c - a), \text{ or} \\
 & a(b - d) = b(a - c); \\
\therefore (3) & a : b = (a - c) : (b - d).
\end{array}
$$

Combining (2) and (3), we have

$$
\begin{array}{ll}
(4) & a + c : b + d = (a - c) : (b - d). \\
 & ad = bc \\
 & \underline{ac = ac} \\
 & a(c - d) = c(a - b); \\
\therefore (5) & a : c = (a - b) : (c - d).
\end{array}
$$

Combining (1) and (5), we have

(6) $(a + b) : (c + d) = (a - b) : (c - d)$.

$$\frac{a}{b}=\frac{c}{d}=\frac{mc}{md}=\frac{\frac{c}{m}}{\frac{d}{m}};$$

$$\therefore\ (7)\ a:b=mc:md,\ \text{and}\ a:b=\frac{c}{m}:\frac{d}{m}.$$

THEOREM VI.

If $a:b=c:d$, then

$$a^2:b^2=c^2:d^2.$$

$$\frac{a}{b}=\frac{c}{d}.\quad \text{Multiplying by}$$

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{a^2}{b^2}=\frac{c^2}{d^2},\ \text{or}\ a^2:b^2=c^2:d^2.$$

The pupils should assume a proportion, and make every possible change upon it, and also combine it with other proportions, each time deducing a theorem by translating the result into words. We shall now apply some of the common principles of proportion to geometric figures.

SECTION XII.—SIMILARITY.

1. **Homologous** sides are opposite equal angles in mutually equiangular figures.

Ques.—Are mutually equiangular figures necessarily regular?

2. Two triangles, or other polygons, are called **similar** when they are mutually equiangular and their homologous sides are proportional.

Cor. 1.—Any two equilateral triangles are similar, and their perimeters are to each other as any side of one is to any side of the other.

Cor. 2.—Any two squares are similar. Any two regular polygons (Def. 2, Sec. VII) of the same number of sides are similar.

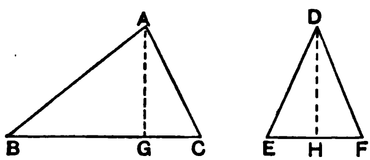
Cor. 3.—Any two circles are similar, and their circumferences are to each other as their radii.

3. The essence of similarity is sameness of shape; of equivalency, is sameness of area. Equality includes similarity and equivalency (Axiom 10).

4. Book I, Theo. XLII, shows that when the diameter of a circle is 1, the circumference is approximately 3.14159. As circles are similar figures, the ratio of the circumference to the diameter is always the same, that is, it is always 3.14159. For convenience, it is usually denoted by π (pi), the initial of the Greek word from which we derive the word perimeter.

THEOREM VII.

If two triangles have the same altitude, their areas are to each other as their bases.



Let the triangles ABC, DEF, have the same altitude; that is, let the perpendiculars AG, DH, be equal. It is to be proved that the area ABC is to the area DEF, as the base BC is to the base EF.

The area ABC is equal to $\frac{1}{2}AG \times BC$ (Cor. 2, Theo. XXIV, Book I); and the area DEF is equal to $\frac{1}{2}DH \times EF$. Therefore,

$$ABC : DEF = \tfrac{1}{2}AG \times BC : \tfrac{1}{2}DH \times EF.$$

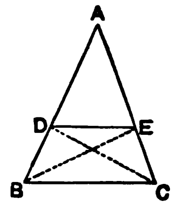
But, from the hypothesis, $\frac{1}{2}$ AG is equal to $\frac{1}{2}$ DH. Hence, we have

$$ABC : DEF = BC : EF.$$

Therefore, if two triangles, etc.

THEOREM VIII.

If a straight line be drawn parallel to the base of a triangle, it will cut the sides proportionally.



In the triangle ABC, let DE be drawn parallel to the base BC. It is to be proved that it cuts AB and AC proportionally.

Join BE and CD. Now, since the triangles ADE, BDE, have the same altitude, namely, the perpendicular distance from E to the line of their bases AB (Def. 4, Sec. V, Book I), their areas are to each other as their bases (Theo. VII); that is,

$$ADE : DBE = AD : DB.$$

Also, because the triangles ADE, CDE, have the same altitude,

$$ADE : CDE = AE : EC.$$

But the first ratio is the same in both proportions; for ADE is common, and the triangles DBE, CDE, having the same base, DE, and the same altitude, namely, the perpendicular distance between the parallels DE and BC, are equivalent (Cor. 2, Theo. XXIV, Book I).

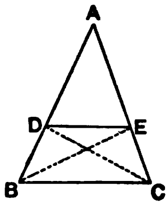
Therefore, by equality of ratios, we have

$$AD : DB = AE : EC.$$

Or, by alternation (Theo. II),

$$AD : AE = DB : EC.$$

Hence, if a straight line, etc.



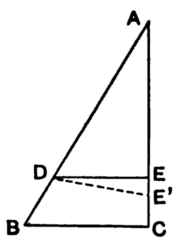
Cor. 1.—By composition (Theo. V), we get from the last proportion,

$$AD + DB : AE + EC = AD : AE;$$

that is, $AB : AC = AD : AE$.

Cor. 2.—If DE is not parallel to BC, but takes some other direction, as DF, it is evident that it does not cut the other sides proportionally.

Cor. 3.—Conversely, if a straight line cut two sides of a triangle proportionally, it will be parallel to the third side, or base.



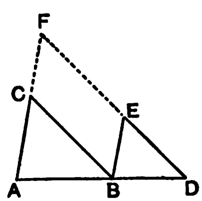
Suppose that DE cut AB and AC proportionally. If it is not parallel to the base, some other line drawn through D, as DE′, is parallel.

On this last supposition prove CE = CE′. Conclusion contradicts what definition?

If no other line than DE through D can be parallel, what conclusion?

THEOREM IX.

Two triangles which are mutually equiangular are also similar.



Let ABC, BDE, be two triangles having the angles at A, B, and C, respectively equal to the angles at B, D, and E. It is to be proved that these triangles are similar.

Let them be so placed that their homologous sides AB, BD (Def. 1, Sec. XII), shall form one straight line. Produce AC and DE till they meet. Then,

because the angle DBE is by hypothesis equal to the corresponding inner angle BAC, the lines BE and AC are parallel (Cor. 1, Theo. III, Book I); and because the angle ABC is equal to the corresponding inner angle BDE, the lines BC and DE are parallel; hence, CBEF is a parallelogram. Now, in the triangle ADF we have (Theo. VIII),

$$DB : BA = DE : EF.$$

But EF is equal to BC (Theo. XX, Book I). Hence,

$$DB : BA = DE : BC.$$

By a like construction, it may be proved in the same way that the other homologous sides are proportional. The two triangles are consequently similar (Def. 2, Sec. XII).

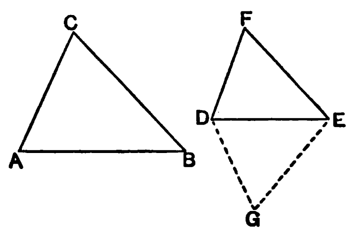
Therefore, two triangles, etc.

Cor.—If two triangles have two angles of the one respectively equal to two angles of the other, they are similar; for in that case the third angles are necessarily equal (Theo. X, Book I).

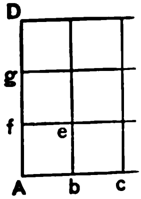
THEOREM X.

Two triangles having the sides of the one successively proportional to the sides of the other, are similar.

Let ABC, DEF, be two triangles having $AB : DE = BC : EF = CA : FD$. It may be shown that they are similar.



On DE describe the triangle DGE, having the angles EDG, DEG, respectively equal to the angles A and B. Then will



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$$DB : BA = DE : EF.$$

EF is equal to BC (Theo. XX, Book I). Hence,

$$DB : BA = DE : BC.$$

...like construction, it may be proved in the same ...at the other homologous sides are proportional. ...two triangles are consequently similar (Def. 2, ...XII).

...efore, two triangles, etc.

...—If two triangles have two angles of the one ...tively equal to two angles of the other, they are ...r; for in that case the third angles are neces... ...y equal (Theo. X, Book I).

THEOREM X.

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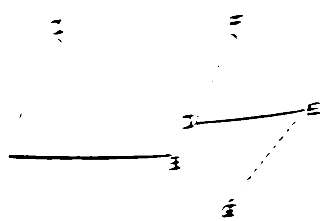
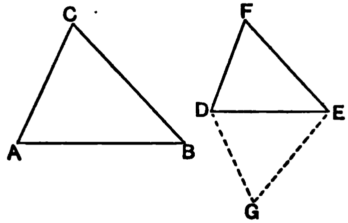


... A and B ...



the triangles ABC and DEG be similar (Cor., Theo. IX); and we shall have (Def. 2, Sec. XII)

AB : DE = BC : EG.

By hypothesis, we have

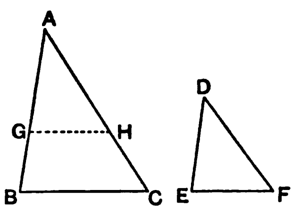
AB : DE = BC : EF.

Hence, EG is equal to EF. In the same manner it may be shown that DG is equal to DF. Therefore, the two triangles DGE, DFE, are mutually equilateral, and consequently equal (Cor. 1, Theo. XIX, Book I). But ABC is by construction similar to DGE; hence, it is also similar to DFE.

Therefore, two triangles, etc.

THEOREM XI.

Two triangles having an angle of the one equal to an angle of the other, and the sides about those angles proportional, are similar.



Let the two triangles ABC, DEF, have the angle A equal to the angle D, and the sides AB, AC, proportional to the sides DE, DF. Then will these triangles be similar.

Take AG equal to DE, and AH to DF; also, join GH. Then the triangles AGH, DEF, having two sides and the included angle of the one equal to two sides and the included angle of the other, are equal throughout (Theo. XII, Book I). Now, by hypothesis,

AB : DE = AC : DF. Therefore, AB : AG = AC : AH.

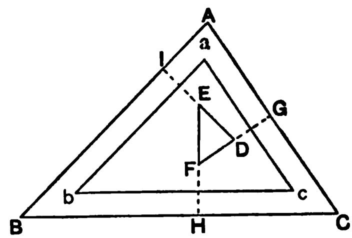
Hence, it follows that GH is parallel to BC (Cor. 3, Theo. VIII). Consequently, the angle AGH is equal to the angle ABC (Theo. III, Book I), and AHG to ACB. Consequently, also (Theo. IX), the triangle AGH, or its equal DEF, is similar to ABC.

Therefore, two triangles, etc.

Cor.—If a triangle, as AGH, be cut off from another triangle, ABC, by a straight line parallel to the base, the two triangles will be similar.

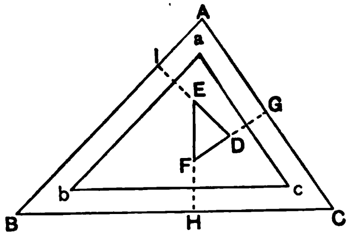
THEOREM XII.

Two triangles having their sides respectively parallel or perpendicular to each other, are similar.



First, let the triangles ABC, *abc*, have their sides respectively parallel. Then, because the sides containing the angle A are parallel to the sides containing the angle *a*, these angles are equal (Cor. 2, Theo. III, Book I); and for the same reason the angles B and *b* are equal, also C and *c*. Hence the two triangles are mutually equiangular, and consequently similar (Theo. IX).

Secondly, let the triangles ABC, DEF, have their sides respectively perpendicular to each other. Produce the sides of DEF to the points G, H, I. Now, the sum of all the angles of the quadrilateral AIDG is equal to four right angles (Theo. XXVII, Book I). But AID and AGD are right angles by hypothesis. Hence, IDG and IAG must be together equal to two right angles. But IDG and EDF are also together

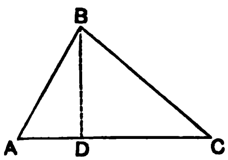


equal to two right angles (Theo. I, Book I). Therefore, subtracting equals from equals, we have the angle IAG equal to the angle EDF. In the same manner it may be proved that the angle ACB is equal to the angle DFE, and CBA to FED. Hence, the two triangles are mutually equiangular, and consequently similar.

Therefore, two triangles, etc.

THEOREM XIII.

A perpendicular let fall from the right angle upon the hypotenuse of a right-angled triangle divides it into two triangles similar to the whole and to each other.



Let ABC be a triangle right-angled at B; and let BD be perpendicular to the hypotenuse AC. It is to be proved that ABD and BCD are similar to ABC and to each other.

The triangles ABD, ABC, have a right angle in each, and the angle A common; hence, they are similar (Cor., Theo. IX). The triangles BCD, ABC, have also a right angle in each, and the angle C common; and hence they are similar. Therefore, also, the triangles ABD, BCD, being both similar to the same triangle, are similar to each other,

Hence, a perpendicular, etc.

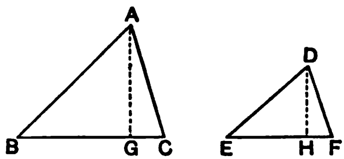
Cor. 1.—By similarity of the triangles ADB, BDC, we have AD : DB : : DB : DC; that is, *the perpendicular from*

the right angle is a mean proportional between the parts of the hypotenuse.

Cor. 2.—Each side about the right angle is a mean proportional between the hypotenuse and the adjacent part.

THEOREM XIV.

The areas of similar triangles are to each other as the squares described on their homologous sides.



Let ABC, DEF, be two similar triangles of which the angles A and B are respectively equal to the angles D and E. It may be shown that their areas are to each other as AB^2 is to DE^2 (Def. 1, Sec. XII).

Draw the perpendiculars AG and DH. Then, the triangles ABG, DEH, having a right angle in each, and the angles B and E equal, are similar (Cor., Theo. IX), and we have

$$GA : HD = AB : DE.$$

But, by similarity of ABC and DEF, we have

$$BC : EF = AB : DE.$$

Multiplying corresponding terms, and dividing the terms of the first couplet by 2,

$$\frac{BC \times GA}{2} : \frac{EF \times HD}{2} = AB^2 : DE^2.$$

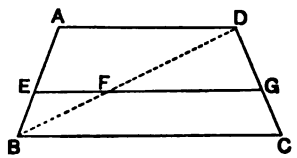
But the first two terms of this proportion represent the areas of the triangles ABC and DEF (Cor. 2, Theo. XXIV, Book I).

Therefore, the areas, etc.

Cor.—The areas of similar triangles are to each other as the squares upon their altitudes, or upon any other pair of homologous lines.

THEOREM XV.

If between the two parallel sides of a trapezoid a third parallel be drawn, it will cut the other sides proportionally.



In the trapezoid ABCD, let EG be a third parallel to AD and BC. Then will BE be to EA as CG is to GD.

Draw the diagonal BD. Now, because EF is parallel to the base AD of the triangle BAD, we have (Theo. VIII)

$$BE : EA = BF : FD;$$

and, because FG is parallel to the base BC of the triangle DBC,

$$CG : GD = BF : FD$$

Therefore, by equality of ratios,

$$BE : EA = CG : GD.$$

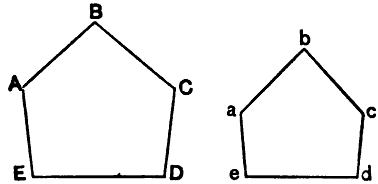
Hence, if between, etc.

Cor. 1.—*If the third parallel bisects the oblique sides, it is equal to one half the sum of the parallel sides.* For, by similarity of the triangles BEF, BAD (Cor., Theo. XI), if BE be one half of BA, EF will be one half of AD; and in the same manner it may be shown that FG will be one half of BC; consequently, the whole line EG will be one half the sum of AD and BC.

Cor. 2.—The area of the trapezoid (Theo. XXV, Book I) is equal to its altitude multiplied by EG, which is its mean breadth.

THEOREM XVI.

The perimeters of similar polygons are to each other as their homologous sides.



Let ABCDE, *abcde*, be two similar polygons, having the angles A, B, C, etc., respectively equal to the angles *a*, *b*, *c*, etc. Then will their perimeters be to each other as any two homologous sides AB and *ab*.

By similarity of the two polygons (Def. 2, Sec. XII), we have

$$AB : ab = BC : bc = CD : cd = DE : de = EA : ea.$$

And therefore, by composition (Theo. V),

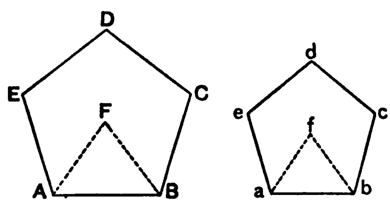
$$AB+BC+CD+DE+EA : ab+bc+cd+de+ea = AB : ab.$$

But the first two terms of the last proportion represent the perimeters of the two polygons (Def. 3, Sec. VII, Book I).

Therefore, the perimeters, etc.

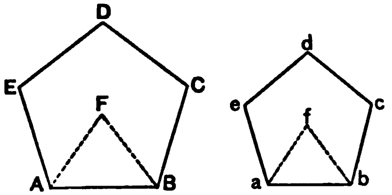
THEOREM XVII.

The areas of two regular polygons of the same number of sides are to each other as the squares of their sides.



Let ABCDE, *abcde*, be two regular polygons of the same number of sides; for example, five. The area of the former is to the area of the latter as AB^2 is to ab^2.

Since the two polygons have the same number of sides and angles, it is evident (Theo. XXVII, Book I) that the sum of all the angles of ABCDE is equal to the sum of all the angles of *abcde*, and consequently



that each angle of ABCDE is equal to each angle of *abcde* (Def. 2, Sec. VII).

Bisect the angles A and B by AF and BF, and the angles *a* and *b* by *af* and *bf*. Now, since the triangles ABF, *abf*, have two angles of the one equal to two angles of the other, they are similar (Cor., Theo. IX); hence,

$$ABF : abf = AB^2 : ab^2 \text{ (Theo. XIV).}$$

Multiplying first couplet by 5,

$$5\,ABF : 5\,abf = AB^2 : ab^2.$$

But five times ABF is the area of ABCDE (Theo. XXIX, Book I), and five times *abf* is the area of *abcde*. Therefore, the areas, etc.

EXERCISES.

1. Prove that two parallelograms of the same altitude are to each other as their bases.

2. Prove that, if two isosceles triangles have their vertical angles equal, they are similar.

3. If the area of a regular pentagon whose side is 1 is 1.72, what is the area of a regular pentagon whose side is 5?

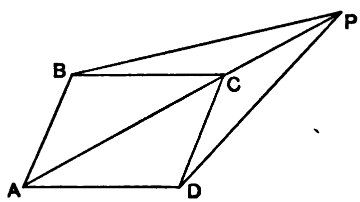
4. In two similar polygons, if a side of one is 7, and the homologous side of the other 13, and if the perimeter of the former is 39, what is the perimeter of the latter?

5. If a quadrilateral is described about a circle, the angles subtended at the center by any two opposite sides are together equal to two right angles.

6. ABCD is a parallelogram, and P any point in the diagonal AC produced. Show that the triangles BCP and DCP are equivalent.

ABC : BCP = AC : CP (?).

ADC : DCP = AC : CP. Complete.



SECTION XIII.—OF CIRCLES.

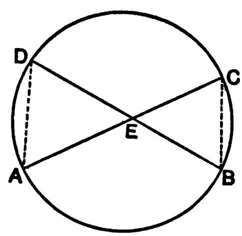
DEFINITIONS.

1. Two **arcs** of circles are called *similar* when they are equal parts of the circumferences to which they belong.

2. Two **sectors** are called *similar* when their arcs are similar.

THEOREM XVIII.

If two chords intersect in a circle, the product of the parts of the one is equal to the product of the parts of the other.

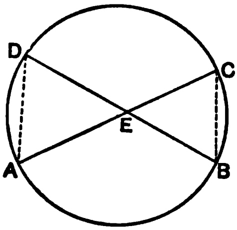


In the circle ABCD let the chords AC, BD, intersect each other in E. It is to be proved that $AE \times EC = BE \times ED$.

Join AD and BC. Now, in the triangles ADE, BCE, the vertical angles AED and BEC are equal (Theo. II, Book I); also, the angles A and B are equal, being both measured by half the same arc, DC (Cor. 1, Theo. XXXIV, Book I);

hence, the third angles are equal, and the two triangles are similar (Theo. IX), and we have

$$AE : BE = ED : EC.$$



Then, multiplying extremes and means, we get

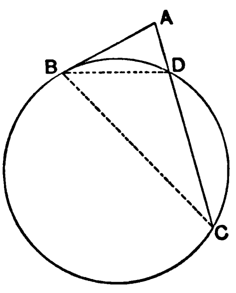
$$AE \times EC = BE \times ED.$$

Therefore, if two chords, etc.

Cor.—If through a fixed point within a circle any chord is drawn, the product of its two segments is constant.

THEOREM XIX.

If from a point without a circle a tangent and a secant be drawn, the tangent will be a mean proportional between the secant and its external part.



Let AB be a tangent and AC a secant to a circle. Then will

$$AC : AB = AB : AD.$$

Join BC and BD. Now, the angle DBA, contained by a tangent and chord, is measured by half the arc DB (Theo. XXXV, Book I); and the angle C, being an inscribed angle, is measured by half the same arc (Theo. XXXIV, Book I); therefore, these two angles are equal. But the angle A is common to the two triangles BAD and BAC. Hence, these triangles have two angles of the one equal to two angles of the other, and are consequently similar (Cor., Theo. IX). Therefore,

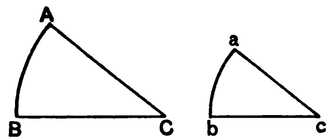
$$AC : AB = AB : AD.$$

Cor.—If from a point without a circle a secant is drawn, the product of the whole secant and its external part is constant.

Sch.—As the distance from the center to a secant increases, the two points where it cuts the arc approach each other. When they coincide, the secant becomes a tangent.

THEOREM XX.

Similar arcs in different circles subtend equal angles at the centers.



Let AB and *ab* be similar arcs in two circles whose centers are C and *c*. It may be shown that the angles at C and *c* are equal.

Whatever portion AB is of the circumference to which it belongs, the angle C is the same portion of four right angles (Sch., Theo. XXXII, Book I); and whatever portion *ab* is of the other circumference, the angle *c* is the same portion of four right angles. But AB is the same portion of the first circumference that *ab* is of the second (Def. 1, Sec. XII). Hence, the angles C and *c* are equal portions of four right angles; hence, also, they are equal to each other.

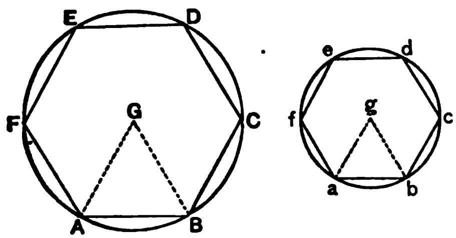
Therefore, similar arcs, etc.

Cor.—The sectors ABC, *abc*, are similar (Def. 2, Sec. XII). Hence, the sides of similar sectors contain equal angles. Hence, also, similar sectors are contained an equal number of times in the circles to which they belong.

Sch.—Since a degree is $\frac{1}{360}$ of the circumference in which it is taken (Sch., Theo. XXXII, Book I), it follows that degrees in unequal circumferences are similar but not equal arcs.

THEOREM XXI.

The circumferences of two circles are to each other as their diameters.



Let ACE and *ace* be two circles having two similar regular polygons inscribed, one in each. Draw the radii AG, BG; *ag*, *bg*. Now, the angles G and *g*, being subtended by similar arcs, AB and *ab*, are equal (Theo. XX); and the sides containing them are by construction proportional; therefore, the two triangles AGB, *agb*, are similar (Theo. XI), and we have

$$AB : ab = AG : ag = 2\,AG : 2\,ag.$$

But the perimeter ABCDEF is to the perimeter *abcdef*, as AB is to *ab* (Theo. XVI), and consequently as 2 AG is to 2 *ag*. Now, if the number of sides of the similar polygons be indefinitely increased, the perimeters will ultimately coincide with the circumferences. Therefore,

$$\text{Circumference ACE} : \text{circumference } ace = 2\,AG : 2\,ag.$$

But 2 AG and 2 *ag* represent the diameters of the two circles (Def. 2, Sec. VIII, Book I).

Therefore, the circumferences, etc.

Cor. 1.—Representing the diameter of any circle by D, and its circumference by C, and remembering that the circumference of a circle whose diameter is 1 is 3.14159 (Sch., Theo. XLII, Book I), we have, by the above theorem,

$$1 : D = 3.14159 : C.$$

Multiplying extremes and means, we get

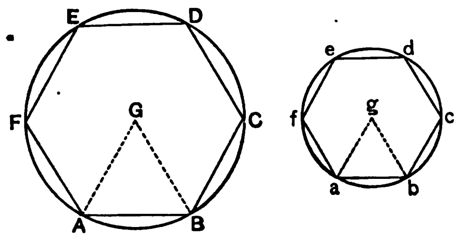
$$C = 3.14159 \times D. \quad \text{That is,}$$

The circumference of any circle is equal to its diameter multiplied by 3.14159, or $C = 2R\pi$.

Cor. 2.—The diameter is equal to the circumference divided by 3.14159, or $D = \frac{C}{\pi}$.

THEOREM XXII.

The areas of two circles are to each other as the squares of their diameters.



The same construction being used as in the last theorem, it may be shown as before, that $AB : ab = 2AG : 2ag$.

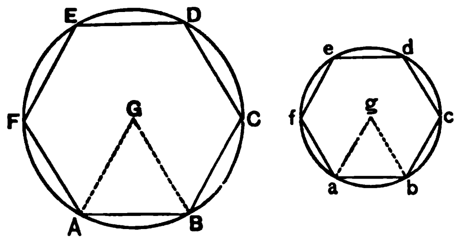
Hence, by squaring all the terms (Theo. VI, Book II),

$$AB^2 : ab^2 = (2\,AG)^2 : (2ag)^2.$$

But the area of the polygon ABCDEF is to the area of the polygon *abcdef*, as AB^2 is to ab^2 (Theo. XVII), and, consequently, as $(2AG)^2$ is to $(2ag)^2$. Now, if the number of sides of the two polygons be indefinitely increased, their areas will ultimately coincide with the areas of the circles. Hence, the area of the circle ACE is to the area of the circle *ace* as $(2AG)^2$ is to $(2ag)^2$.

That is, the areas of two circles, etc.

Cor.—Representing the diameter of any circle by D, and its area by A, and remembering that the area of a circle whose diameter is 1 is .7854 (Cor., Theo. XLII, Book I), we have



$1^2 : D^2 = .7854 : A.$

But 1^2 is 1. Therefore, multiplying extremes and means, we get $A = .7854 \times D^2$. That is,

The area of any circle is equal to the square of its diameter multiplied by .7854.

EXERCISES.

1. If a secant to a circle be 12 feet, and its external part 3 feet, what will be the length of a tangent to the circle, drawn from the same point?

2. If the diameter of a circle be 24, what will be the length of its circumference?

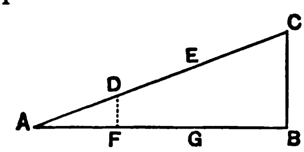
3. If the circumference of a circle is 3924, what is the length of its diameter?

4. The diameter of a circle is 16: what is its area?

5. Prove that the areas of similar sectors are to each other as the squares of their radii.

6. Prove that the area of a circle equals the square of its radius multiplied by 3.14159.

7. To divide a straight line into any number of equal parts.

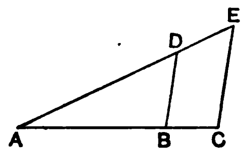


Let AB be the given straight line, which it is required to divide into a certain number of equal parts; for example, three.

From A, draw any straight line AD; and lay off DE and EC in the same direction,

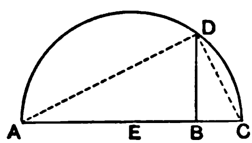
each equal to AD. Join CB, and through D draw DF parallel to CB. Now, AD : AC = AF : AB (Cor. 1, Theo. VIII). But AD is one third of AC, by construction; therefore, AF is one third of AB. Now, lay off FG equal to AF, and it is evident that AB will be divided into three equal parts

8. To find a fourth proportional to three given straight lines.



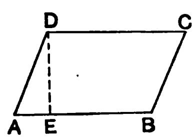
Draw two straight lines containing any angle A. Make AB, BC, and AD respectively equal to the three given lines, and join BD. Through C draw CE parallel to BD, and produce AD to meet it in E. Now, AB : BC = AD : DE (Theo. VIII). Hence, DE is the fourth proportional required. Define fourth proportional.

9. To find a mean proportional between two given straight lines.

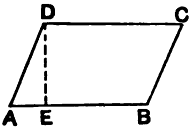


Lay off, in one straight line, AB and BC, respectively equal to the two given lines. Find E, the middle point of AC. Then, from E as a center, with EA or EC as radius, describe a semicircle. At B erect BD perpendicular to AC, and join AD and DC. Now, since the angle ADC is inscribed in a semicircle, it is a right angle (Cor. 2, Theo. XXXIV, Book I). Hence, BD is a mean proportional between AB and BC (Cor. 1, Theo. XIII).

10. To describe a square that shall be equivalent to a given parallelogram.



Let ABCD be the given parallelogram. Between its base AB and its altitude ED, find a mean proportional, which denote by M.



Then, since $AB : M = M : ED$, we have $AB \times ED = M^2$.

But $AB \times ED$ represents the area of the parallelogram (Cor. 1, Theo. XXIV, Book I). Therefore, the square described on M is the square required.

Cor.—It is evident that, if we find a mean proportional between the base of a triangle and half its altitude (Cor. 2, Theo. XXIV, Book I), we shall have the side of a square equivalent to the triangle.

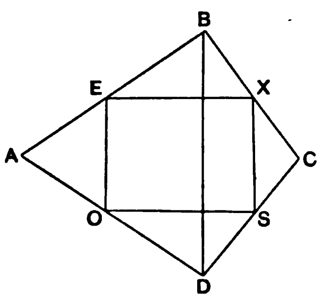
11. To construct a triangle that shall be similar to a given triangle.

12. On a given straight line, to describe a rectangle that shall be equivalent to a given square.

At the extremity of the given line AB, erect a perpendicular BD equal to the side of the given square. Complete. See Prob. IX.

13. A line joining the middle points of two sides of a triangle is parallel to the third, and equal to the half of it.

14. Bisect the sides of a quadrilateral, join the points of bisection, and prove that the inscribed figure is a parallelogram.

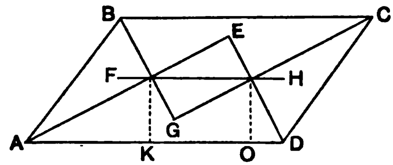


Prove EO equal and parallel to SX.

Ques.—What is the conclusion?

Cor.—Determine the ratio between the areas of the parallelogram and quadrilateral.

15. Bisect the angles of a parallelogram, and prove that the bisectors form a rectangle whose diagonals are parallel to the sides of the parallelogram.

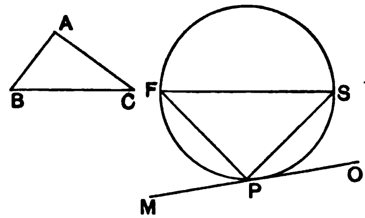


The angles A and D are supplementary (?).

∴ E is a right angle (?). Similarly, F, G, and H.

The point F is equally distant from AD and BC (Sec. V, Exercise 7). Also, the point H is equally distant from AD and BC. FH is parallel to AD (Theo. XXI, Book I).

16. In a given circle, to inscribe a triangle similar to a given triangle.

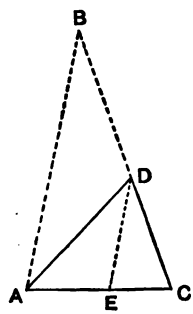


Draw the tangent MPO. Make the angle SPO = C and FPM = B. Show that the triangles FPS and BAC are similar (Theorem XXXVI, Book I).

17. In a given triangle, to find the side of an inscribed square.

Draw the figure, and call the base b and the altitude a. Let x = side of the square, and find its value in terms of a and b, and construct.

18. To divide the base of a triangle into segments proportional to the adjacent sides.



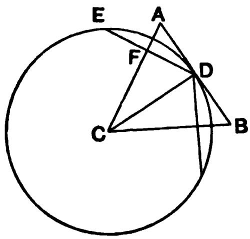
Bisect the vertical angle D by the line DE, and AE and EC will be the required segments.

Draw AB parallel to DE, the bisecting line.

Ques.—What kind of a triangle is ABD?

DC : EC = DB : EA (Theo. VIII Book II). Complete.

19. Determine the area of an inscribed, and also of a circumscribed, hexagon in terms of radius.



ED is a side of the inscribed hexagon; and AB, of the circumscribed hexagon.

CD, or R, $=$ ED (?).

$$FC = \frac{R}{2}\sqrt{3}.$$

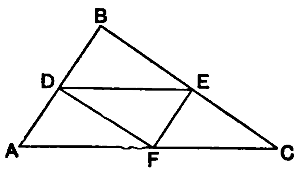
$CD^2 = AC \times FC$ (Cor. 2, Theo. XIII).

$$\therefore R^2 = AC \times \frac{R}{2}\sqrt{3}.$$

$AC = 2\ AD$ (?)

Prove that the two hexagons are to each other as 3 is to 4.

20. To divide a triangle into four equal triangles.



Connect the centers of the three sides by the lines DE, DF, and EF.

The triangles DBE and ABC are similar (Theo. XI, Book II). Therefore (Def. 2, Sec. XII),

$$BA : BD = AC : DE.$$

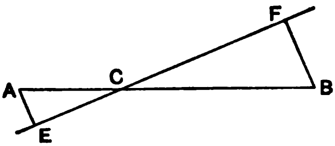
But $BD = \frac{BA}{2}$. $\therefore DE = \frac{AC}{2}$.

DE is parallel to AC (Theo. VIII, Cor. 3, Book II).

$\therefore$ ADEF is a parallelogram (?). See Theorems XX and XXI, Book I. Complete.

21. The difference between the side and the diagonal of a square being given, construct the square.

22. On any line, as AB, select the point C, and show that the perpendiculars from A and B, drawn to any straight line through C, have to one another a constant ratio.

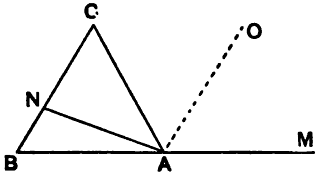


EF is any straight line cutting AB at C. AE and BF are the perpendiculars.

Prove AE ÷ FB = AC ÷ CB, in any position of EF not coinciding with AB.

23. By construction, CB is equal to CA.

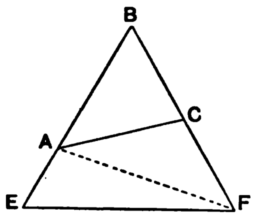
By construction, AN is equal to CA.



Prove the angle NAM equal to 3B. Draw AO parallel to BC.

ANC = ACN = CAO. ANC = CBA + BAN. Complete the proof.

24. Two triangles which have an angle of the one equal to an angle of the other, are to each other as the products of the sides including the equal angles.

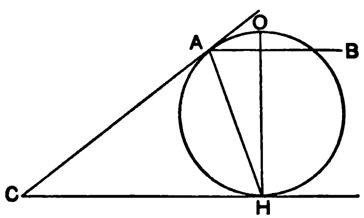


See Theo. VII.

BAC : BAF = BC : BF (?).
BAF : BEF = BA : BE (?).
BAC : BEF = BC × BA : BF × BE (?).

25. CA and CH are tangents.

OH is a diameter.

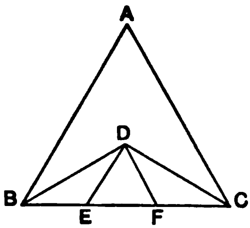


Prove AHO $= \frac{C}{2}$.

Sch.—Write out the theorem.

26. The base of a triangle is 8: $\frac{3}{4}$ of the triangle are cut off toward the base by a line parallel to it; how long is the cutting line?

27. Trisect a given straight line.

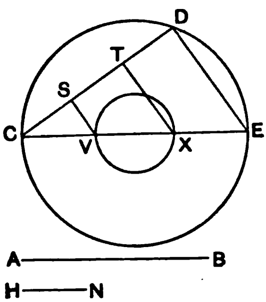


On the given line BC, construct an equilateral triangle. Bisect the angles B and C, and from D draw DE and DF parallel to AB and AC.

Prove BE = EF = FC.

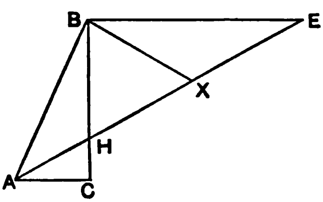
Ques.—What kind of a triangle is BED? EDF?

28. Describe a circle concentric with another circle, and equal to $\frac{1}{9}$ of it.



Assume HN less than one third of the diameter, and take AB = 3HN. From C draw a chord equal to AB, and trisected into parts each equal to HN. From D draw DE, and, from the points of trisection, S and T, draw lines parallel to DE. The diameter CE will be trisected, and a circle drawn with VX as a diameter is the required circle (Theo. XXII, Book II).

Cor.—Substitute $\frac{1}{4}$ for $\frac{1}{9}$, and give the solution. With what fractions will this method apply? Why is the length of CD not important?



29. ACB is right angled at C.

BE is parallel to AC.

AE is so drawn that HE = 2AB.

Prove EAC = $\frac{1}{3}$BAC.

Draw BX to the center of HE.

Sch.—As HE must be *assumed* equal to twice AB,

this is not a general plan for the trisection of an acute angle.

30. The breadth and length of a rectangular field are to each other as 3 and 4. To inclose the field cost $392. What would it cost to divide the field diagonally with the same kind of a fence?

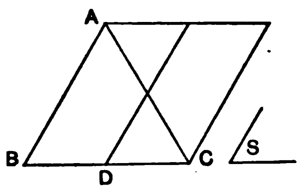
31. Construct a parallelogram equal to a given triangle, and having one of its angles equal to a given angle.

The triangle is BAC, and S is the given angle.

D is the center of BC.

Construct an angle at D equal to S. It may differ from B.

Complete.



32. Prove that the diagonal of a rectangle is the longest line whose extremities are in the sides of the rectangle.

33. The opposite angles of a quadrilateral which can be inscribed in a circle are supplementary.

34. Two secants drawn from the same point are to each other inversely as their external parts. That is, one diagonal is to the other, as the external part of the second is to the external part of the first.

35. From the top of a mountain 2 miles high, the horizon out at sea was distant 126 miles. What is the diameter of the earth?

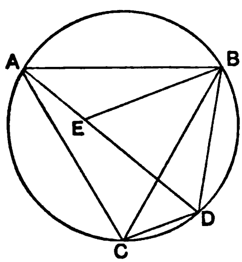
36. When two circumferences intersect, the straight line joining their centers bisects at right angles their common chord.

37. From two points in the circumference of a circle two straight lines are to be drawn to a point in a tangent to that circle. What point in the tangent must be chosen so that the lines shall make the greatest angle possible?

38. From a point without a circle a line, 8 feet in length, is drawn tangent to the circle, and a line 4 feet is drawn meeting the circle at its nearest point. What is the area of the circle?

39. The radius of a circle is 12 feet. Find the area of one of the segments cut off by the side of an inscribed square.

40. Join any point on the circumference of a circle which is circumscribed about an equilateral triangle with the three vertices. The middle one of these lines is equal to the sum of the other two.



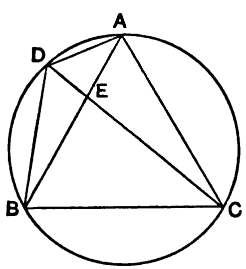
Let ABC be an equilateral triangle. Then is AD = CD + DB. Draw BE, making the angle DBE = BDE.

The triangles ABC and EBD are similar (?). BD = ED (?).

The triangles AEB and CDB are equal (?). CD = AE (?).

$\therefore$ AE + ED = CD + BD, or AD = CD + BD.

41. Another demonstration of No 40.



The triangles DEA and BEC are similar (?). Likewise DEB and AEC are similar. Also EBC and DBC (?).

(1) DA : BC = AE : EC.

(2) DB : AC = BE : EC.

DA $\times$ EC = BC $\times$ AE.

DB $\times$ EC = AC $\times$ BE.

Adding,

(3) (DA + DB) EC = BC^2.

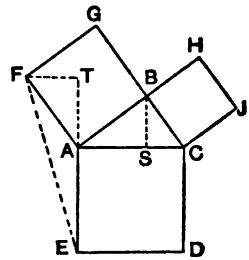
(4) DC : BC = BC : EC.

DC $\times$ EC = BC^2. $\therefore$ (DA + DB) EC = DC $\times$ EC.

$\therefore$ DA + DB = DC.

Sch.—Assume arc AD = arc BD, and prove Ex. 40.

42. Draw a figure as in Theo. XXVI, Book I. Connect the points E and F, G and H, and J and D. Prove the triangles FAE, GBH, and JCD each equivalent to ABC.



The side EA is prolonged, and FT let fall upon the base produced.

FT = BS (?).

Examine the triangles FTA and ABS.

43. Draw a line parallel to the base of a triangle, which shall equal the sum of the lower segments of the sides.

Bisect the angles at the base.

44. In any triangle, given one angle, a side adjacent to the given angle, and the difference of the other two sides, to construct the triangle.

First, construct a triangle of the given side and difference, with the given angle included.

45. The vertical angle of an oblique-angled triangle inscribed in a circle, is greater or less than a right angle, by the angle contained between the base and the diameter drawn from the extremity of the base.

46. Draw semicircles upon the three sides of a right-angled triangle, and the area of the crescents thus formed will equal that of the triangle.

47. Draw a diameter through P, a point within a circle, and the two segments of the diameter will measure the least and the greatest distance from P to the circumference.

48. If from the middle point P, of an arc NS, the chords PD and PC are drawn cutting the chord NS in A and B, the opposite angles of ABCD are supplementary.

49. Through the point P, without a given angle,

draw a line cutting the sides of the angle, so that the triangle formed shall have a given perimeter.

50. From the centers of two opposite sides of any quadrilateral, draw lines to the centers of the diagonals, and the figure thus formed is a parallelogram.

51. Upon AC, the diagonal of a square ABCD, mark off AP equal to AB, and draw PO perpendicular to the diagonal and terminating in the side, and prove $BO = OP = PC$ (Sec. X, Ex. 17).

52. Two circles touch each other at T, and have a common tangent AB. The chords AT and BT form a right angle. Connect the centers, and draw radii to A and B (Theorems XXXI and XXXV, Book I).

53. The square described upon a line is equal to the squares upon any two segments of the line increased by twice the rectangle contained by those segments.

54. Two chords of a circle intersect at right angles. Prove that the squares of the four segments are together equal to the square of the diameter. (Theo. XIII, Cor. 1, Book II).

Sch.—Upon review, pupils should at times be incited to disregard the methods suggested in the lists of Exercises, and search for others. Read the preface carefully before beginning the review.

SOLID GEOMETRY.

BOOK III.

SECTION XIV.—PLANES AND LINES.

DEFINITIONS.

1. In **solid geometry** the magnitudes treated of are not limited to one plane.

2. A **plane** has already been defined to be a surface such that, if any two of its points be connected by a straight line, the line will lie wholly within the surface.

3. A straight line is **perpendicular to a plane** when it is perpendicular to every straight line of that plane which it can meet. The plane is also **perpendicular to the line.**

4. The line in which two planes cut one another is called their **intersection.**

Cor. 1.—*The intersection of two planes is a straight line;* for, if any two points in the intersection be taken, the straight line joining these points must be in both planes (?).

Cor. 2.—Two planes have no common point without their intersection (?).

5. The **angle of inclination** of two planes is that

contained by any two straight lines perpendicular to their intersection, one in each plane.

If the line in each plane is perpendicular to the other plane, the two planes are said to be **perpendicular** to each other, the angle of inclination being a right angle.

6. The **diedral** is the angle formed by two planes which cut one another.

By simply opening a book at various angles, we have examples of acute, right, and obtuse diedrals.

7. Two planes are said to be **parallel** when their intersections by every third plane which can cut them are parallel.

Cor. 1.—*Two parallel planes can never meet;* for, if they were to meet in any point, their intersections by a third plane passing through that point would also meet. By the definition, all such intersections are parallel.

Cor. 2.—Two planes which are parallel to a third are parallel to each other.

Cor. 3.—Two planes which are perpendicular to the same straight line are parallel to each other.

8. The **projection of a point** upon a plane is the foot of a perpendicular let fall from the point to the plane. The **projection of a line** contains the foot of every perpendicular which can be drawn from the line to the plane; that is, it is the **locus** of the projections of all its points.

Cor.—The projection of a straight line upon a plane is a straight line.

Ques.—What exception to this statement?

9. The **inclination of a line to a plane** is the angle it makes with its projection on the plane.

10. The **position** of a plane is determined by its containing three points not in a straight line, or by its containing a straight line and a point without it. The plane may be turned upon the line as an axis until it embraces the given point.

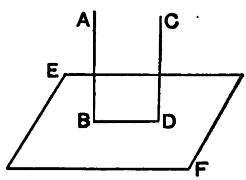
11. The **axis of a circle** is a straight line perpendicular to the circle at its center.

Cor.—Any point in the axis is equally distant from all points in the circumference (?).

Ques.—What is the **locus** of all the points equally distant from three given points?

THEOREM I.

Two straight lines perpendicular to the same plane are parallel.



Let AB and CD be two perpendiculars to the plane EF. It is to be shown that they are parallel. Join BD. Now, AB and CD being perpendicular to the plane, are also perpendicular to the line BD (Def. 3).

Also, a plane perpendicular to EF, and passing through BD, will contain both AB and CD (Def. 5). But straight lines in the same plane, and perpendicular to the same straight line, are parallel (Cor. 2, Theo. IV, Book I.) Therefore, AB is parallel to CD.

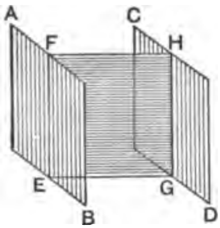
That is, two straight lines, etc.

Cor.—If one of two parallels is perpendicular to a plane, the other is also perpendicular to the plane.

Ques.—Can two lines cut a plane at the same angle and not be parallel?

THEOREM II.

Parallel lines intercepted between parallel planes are equal.



Let FH and EG be any two parallel lines intercepted between two parallel planes, AB and CD. It is to be shown that FH is equal to EG.

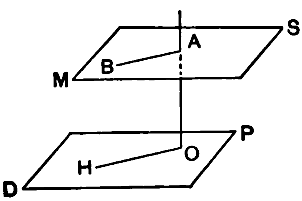
Through the parallels FH and EG let the plane FG pass. Its intersections with the parallel planes AB and CD will be parallel lines (Def. 7); that is, FE is parallel to HG. Hence, FEGH is a parallelogram, and FH is equal to EG (Theo. XX, Book I).

Therefore, parallel lines, etc.

Cor.—The perpendicular distance between two parallei planes is everywhere the same.

THEOREM III.

A straight line perpendicular to one of two parallel planes is perpendicular to the other.



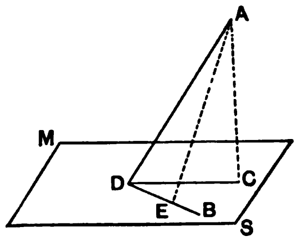
Let MS and DP be parallel planes, and let the straight line AO be perpendicular to MS; then it will be perpendicular to DP.

BA is any line in the plane MS drawn from A. Pass a plane through AO and AB. It will cut DP in some line, as HO, parallel to AB (Def. 7). But BA is perpendicular to AO (Def. 3), and, therefore, HO is perpendicular to AO. But HO is *any* line in the plane DP, therefore, AO is perpendicular to the plane DP (Def. 3).

THEOREM IV.

When an oblique line meets a plane, the angle which it makes with its own projection is the least angle it makes with any line in the plane.

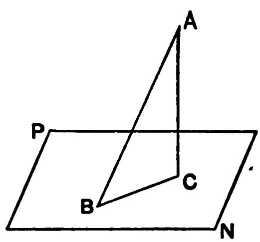
Let DC be the projection of AD upon the plane MS, and DB be any other line in the plane drawn from D, and DE = DC. Noting that AC is the only perpendicular which can be drawn from A to the plane, prove ADC < ADE (Book I, Theo. XVII, Cor.).



THEOREM V.

From a point without a plane there can be only one straight line drawn perpendicular to the plane.

If there can be two, as AB and AC, draw the line BC, and we have the triangle ABC containing two right angles.



Ques.—What false assumption leads to this absurdity?

Cor.—From the point A an infinite number of equal lines can be drawn to the plane PN, and their points of intersection with the plane will form a circle.

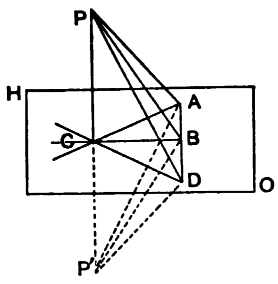
Ques. 1.—In what case would this infinite number be reduced to one?

Ques. 2.—How can a perpendicular to a plane be drawn from a point without the plane?

Ques. 3.—A room is ten feet high. How, by the us; of an inflexible rod eleven feet long, can a point on the floor be found directly beneath a given point on the ceiling (Def. 11, Sec. XIV, Cor.)?

THEOREM VI.

A line, perpendicular to two lines of a plane at their intersection, is perpendicular to the plane.



If PC is perpendicular to two lines, CA and CD, of the plane HO, at their intersection, then will it be perpendicular to *any* other line of the plane, as CB, drawn to its foot; and, therefore (Def. 3, Sec. XIV), be perpendicular to the plane.

Extend PC below the plane till P′C = PC. Draw the line AD cutting CB in B, and draw PA, PB, PD, and P′A, P′B, P′D.

PB is equal to P′B.

For PA = P′A (Book I, Theo. IX).

PD = P′D.

∴ the triangles PAD and P′AD are equal (?).

And the triangles PAB and P′AB are equal (Book I, Theo. XII).

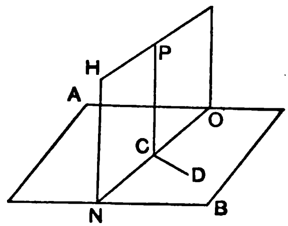
∴ PB = P′B, and, as PC = P′C, the line CB is perpendicular to PP′ (Book I, Theo. XIII, Cor. 5). And PC is perpendicular to the plane HO.

Ques. 1.—How many perpendiculars to the plane HO can be drawn from P?

Ques. 2.—How many parallels to the plane can be drawn through P?

THEOREM VII.

If a line is perpendicular to a plane, every plane passed through the line is also perpendicular to that plane.

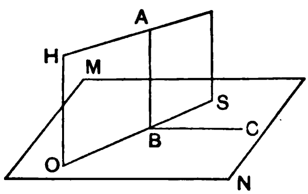


PC is perpendicular to the plane AB. Pass through PC the plane HO. Then will HO be perpendicular to AB.

From C draw CD, in the plane AB, perpendicular to the intersection ON; then will PCD be a right angle, and, as it measures the *angle of inclination* (Def. 5, Sec. XIV), the plane HO is perpendicular to the plane AB, and HO is *any* plane passed through PC.

THEOREM VIII.

If two planes are perpendicular to each other, a straight line in one of them, perpendicular to their intersection, is perpendicular to the other.



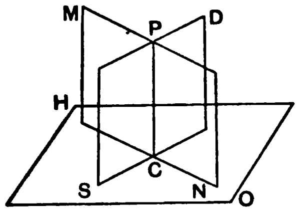
The planes HS and MN are perpendicular to each other. AB is a straight line in HS perpendicular to OS, their intersection. BC is drawn in the plane MN perpendicular to OS at B. Then, ABC is a right angle (?).

The line AB is perpendicular to BS and BC in the plane MN. It is therefore perpendicular to the plane (Theo. VI).

Ques.—How can you draw a line upon the wall, which will be perpendicular to the floor?

THEOREM IX.

If two planes are each perpendicular to a third plane, their intersection is perpendicular to that plane.

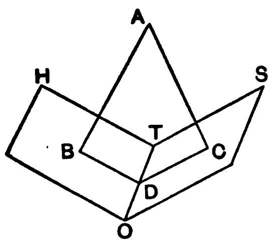


The perpendicular to the plane HO, at the point C, must lie in both the planes, MN and SD(?). Hence, it must be their intersection.

Cor.—Place your pencil perpendicular to your slate, and explain the principle.

THEOREM X.

The angle included by two perpendiculars drawn from any point within a diedral to its faces, is the supplement of the diedral angle.



Let A be a point within the diedral H-OT-S, and AB and AC the perpendiculars to its faces, HO and SO. Then will BAC be the supplement of the diedral angle; or, more strictly, the supplement of the angle which measures the diedral.

Apply Theorems VII and IX, and prove that BDC is the diedral angle. Finish the demonstration.

Cor. 1.—Any two parallel lines lie in the same plane.

Cor. 2.—Any two lines forming an angle lie in the same plane.

Cor. 3.—Through a given line of a plane, only one plane can be passed perpendicular to the given plane.

SECTION XV.—POLYEDRALS.

DEFINITIONS.

1. The divergence of three or more planes from a single point forms a **polyedral,** or **polyedral angle.** If the divergence is of three planes, the angle is called a **triedral.**

The corner of the room, where two walls and the ceiling have a common point, is a triedral.

2. When, as in the example just given, the three diedrals of a triedral are each **right,** the angle is called a **trirectangular** triedral.

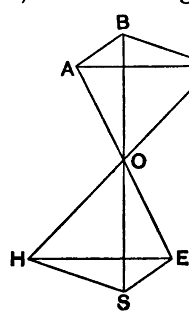
3. An **isosceles** triedral is one which has two equal face angles.

4. An **equilateral** triedral is one which has three equal face angles.

5. Two triedrals are **equal** when, being applied one to the other, they coincide in all their parts.

6. Two triedrals are **symmetrical** if they are composed of parts which, taken one by one, are respectively equal, but are not similarly arranged.

Suppose we have three lines passing through a common point, but not all in one plane; and planes passing through these lines, two and two, as in this figure.



The lines AOE and COH are in the plane of the paper. Conceive the point B as behind that plane, and the point S in front of it.

O-ABC and O-HSE are symmetrical triedrals.

Pupils should analyze the figure, and show that its two parts correspond to the definition of symmetrical.

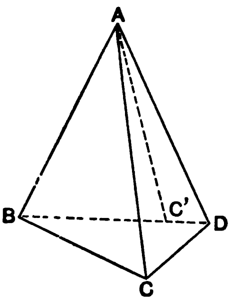
THEOREM XI.

The sum of any two face angles of a triedral is greater than the third face angle.

Let BAD be the greatest of the face angles of the triedral at A. Then, BAC + CAD > BAD.

Make the angle BAC′ = BAC, and AC′ = AC.

Then will the triangles BAC and BAC′ be equal (?).



BC + CD > BD (?). That is BC + CD > BC′ + C′D.

∴ CD > C′D (?).

In the triangles CAD and C′AD the angle CAD > C′AD (Book I, Theo. XVII, Cor.). Add the equal angles BAC and BAC′, and we have BAC + CAD > BAD.

THEOREM XII.

The sum of the three face angles of any triedral is less than four right angles.

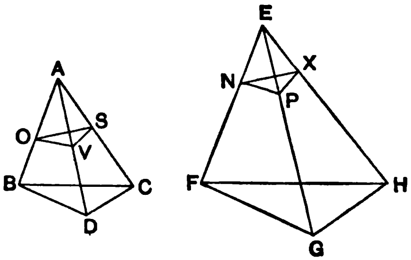
Draw the figure used in the last theorem, omitting AC′.

The sum of the angles of the triangles having a common point at A is *six* right angles (?).

Prove, by Theorem XI, that the sum of the angles ABD + ABC + ACB + ACD + ADC + ADB is greater than DBC + BCD + CDB; that is, greater than *two* right angles. Complete the demonstration.

THEOREM XIII.

Two triedrals having the face angles of the one equal to the face angles of the other, each to each, and similarly arranged, are equal.



By hypothesis, the angles BAD and FEG are equal; likewise DAC and GEH, and BAC and FEH.

It remains to prove the diedrals respectively equal and similarly arranged.

By construction, AS = EX.

OS and VS are drawn perpendicular to AC in their respective planes. NX and PX are drawn under similar conditions.

Hence, the angle OSV measures the inclination of the two planes whose intersection is AC; that is, the diedral D-AC-B; and NXP measures the diedral G-EH-F (Definitions 5 and 6, Sec. XIV).

But the triangle OSV = the triangle NXP; for, OS = NX, and VS = PX (Theo. XVIII, Book I), and OV = NP (Theo. XII, Book I). Hence, the triangles are mutually equilateral and (Theo. XIX, Book I) equiangular.

Therefore the angle OSV = the angle NXP; that is, the diedrals are equal.

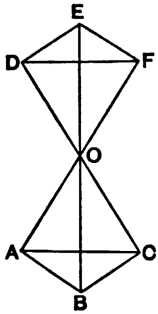
In the same way the diedrals whose edges are AB and EF, and those whose edges are AD and EG, may be proved equal.

Ques.—Will the equality of the plane angles OVS and NPX prove the equality of the diedrals B-AD-C and F-EG-H?

Sch.—Notice that the corresponding faces, the triangles ADC and EGH, etc,, are not necessarily equal; the planes cutting BCD and FHG being used simply to aid the eye to comprehend the triedral angles.

THEOREM XIV.

An isosceles triedral and its symmetrical are equal.



By definition, the triedral O-DEF is the symmetrical of O-ABC. By hypothesis, the angle AOB = BOC.

If the upper part of the figure rotate about the point O till DOF coincides with AOC, prove that the triedrals coincide throughout, and are equal.

Cor.—An isosceles triedral has two equal diedrals.

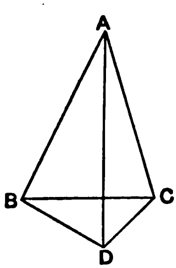
SECTION XVI.—POLYEDRONS AND THE CYLINDER.

Three planes may be conceived which have no intersection. They are parallel each to the others.

Three planes may have one intersection. They pass through a common line, as lines in a plane may pass through a common point.

Three planes may have two intersections. Two of them are parallel, like two opposite walls of the room, and the third cuts them.

Three planes may have three intersections. Two of them may be to each other as the adjacent walls of a room, and the third may cut them like a partition passed through the diagonals of the floor and ceiling; or the third may cut them as the ceiling cuts them, and form a trirectangular triedral. The face angles may not be right angles. In the triedral A, each



of the triangles BAD, BAC, and DAC is a portion of a plane, and three planes have three intersections each passing through A, and each of the face angles may be greater or less than a right angle. It is evident that three planes in no one of these positions enclose a space on all sides; but that, in one case, a fourth plane may complete the enclosure. For example, a plane cutting the intersections AB, AD, and AC, at B, D, and C.

Here we have the smallest number of planes which can bound a portion of space, viz., four.

DEFINITIONS.

1. A **polyedron** is a geometrical solid, or portion of a space bounded by planes, called **faces.**

The intersections of the faces are called **edges;** and of the edges, **vertices.**

2. The **volume** of a polyedron is the number of times it contains some unit of measure. What **kind** of a unit?

3. A **diagonal** of a polyedron joins two vertices not in the same face.

A **diagonal plane** divides a polyedron by passing through two of its edges.

4. A polyedron of four faces is called a **tetraedron;** of five, a **pentaedron;** of six, a **hexaedron,** etc.

Evidently, a polyedron may have any number of sides more than three.

5. A **regular** polyedron has equal polygons for its faces, and its polyedral angles all equal.

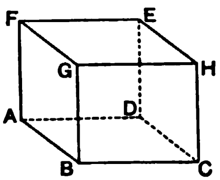
There are but five regular polyedrons.

6. Similar polyedrons are identical in form. They have the same number of faces respectively similar and similarly placed, and their respective polyedrals are equal.

A single brick might be of the same shape as a house. In case it were, its length, breadth, and height, multiplied by a common multiplier, would equal the length, breadth, and height of the house.

7. A **parallelopiped** is a hexaedron the sides of which are each parallel to its opposite.

Cor.—Any face of a parallelopiped is a parallelogram. For the lines GH and BC, being the intersections of two parallel planes, FH and AC, with a third plane, GC, are parallel (Def. 7, Sec. XIV); and in the same way it may be shown that BG and CH are parallel; hence, BCHG is a parallelogram (Def. 2, Sec. VI, Book I).



8. A **right parallelopiped** is one whose edges are perpendicular to the faces. Any other parallelopiped is called **oblique.** When the bases are rectangles, the figure is a **rectangular** parallelopiped.

9. A **cube** is a right parallelopiped whose length, breadth, and height are equal.

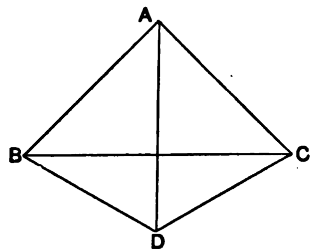
10. Two parallelopipeds, or other polyedrons, are called **equivalent** when they have the same volume.

THEOREM XV.

There is a point equally distant from the vertices of a tetraedron.

The axis of the circle passing through B, A, D, contains all the points equally distant from B, A, D (Def. 11, Sec. XIV). The axis of the circle passing through B, A, C, contains all the points equally distant from B, A, C.

These axes both lie in one plane—that which is perpendicular to BA at its center. The axes can not be parallel (?). They therefore meet; and their point of intersection has the properties of both lines, and is at the same distance from B, A, C, and D.



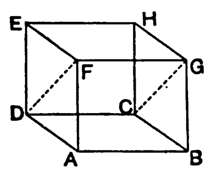
THEOREM XVI.

There is a point in every tetraedron equally distant from the four faces.

Use the last figure, and give demonstration. The argument is similar to that by which we prove that there is a point in every triangle equally distant from the three sides.

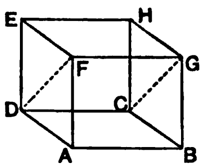
THEOREM XVII.

Any two opposite faces of a parallelopiped are equal.



Let AH be a parallelopiped. It is to be proved that any two opposite faces, as DEFA, CHGB, are equal. Join DF and CG. Now, since EFGH is a parallelogram (Cor., Def. 7), EF is equal to HG (Theo. XX, Book I); and in the same manner it may be shown that DE is equal to CH. But FG and DC, being both equal and parallel to EH (Theo. XX, Book I), must be equal and parallel to each other. Hence, DCGF is a parallelogram (Theo. XXI, Book I), and DF is equal CG. Wherefore, the triangles DEF, CHG, are mutually equilateral,

and consequently equal. But the former triangle is half the parallelogram DEFA (Cor. 1, Theo. XX, Book I); and the latter triangle is half the parallelogram CHGB. Hence, these two parallelograms are equal.



Therefore, any two opposite faces, etc.

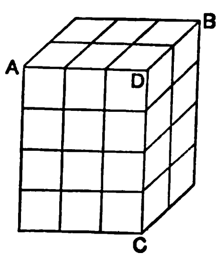
Sch.—Any face of a parallelopiped may be taken as the **base.** We may designate DABC and EFGH as the **lower and upper bases.** The perpendicular distance between the bases is called the **height** or **altitude.**

Cor. 1.—If EB be a right parallelopiped, the edges AF and BG—since, by definition, they are both perpendicular to the face DB—must be perpendicular also to the straight line AB, which they meet in that plane (Def. 3, Sec. XIV). Hence, ABGF, or **any face of a right parallelopiped, is a rectangle.**

Cor. 2.—The faces of a cube are all equal squares.

THEOREM XVIII.

The volume of a right parallelopiped is equal to the area of its base multiplied by its height.



Let ABC be a right parallelopiped. It is to be proved that its volume is equal to the area of its base, AB, multiplied by its height, DC.

Let planes pass through the solid parallel to the three faces AB, BC, CA, dividing the edges DA, DB, DC into parts of equal length. It is evident that the parallelopiped will thus

be divided into a number of equal cubes. Let one of these be taken as the unit of solidity. Now, in the layer next to the base, there will be as many of these solid units as there are corresponding units of area in the base; and there will be as many equal layers as there are corresponding linear units in the height.

Therefore, the volume of a right parallelopiped, etc.

Sch.—If the edges are incommensurable, so that no unit can be found which will be a common measure of all of them, the above theorem still holds true; for, by taking the unit smaller and smaller, the remainder can be made less than any assignable quantity.

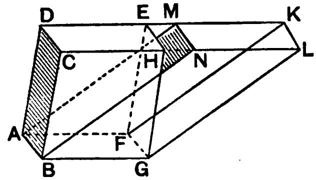
When the linear unit is one inch, the unit of solidity is a cubic inch; when the linear unit is one foot, the unit of solidity is a cubic foot, etc.

Cor. 1.—The volume of a right parallelopiped is equal to the product of its length, breadth, and height.

Cor. 2.—The volume of a cube may be found by raising one of its edges to the third power.

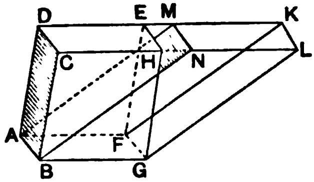
THEOREM XIX.

Parallelopipeds on the same base and between the same parallels are equivalent.



Let AGHD and AGLM be two parallelopipeds on the same base AG, and between the same parallels. BG, CL; AF, DK. It may be shown that they are equivalent.

The parallelograms AC, FH, being opposite faces of a parallelopiped, are equal (Theo. XVII). Also, the

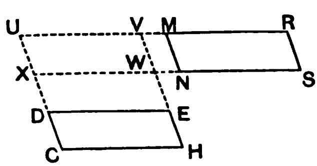


parallelograms DH, ML, being both equal to AG, are equal to each other; hence, if we add EN to both, the parallelograms DN and EL must be equal. It is also evident that the side CN is equal to the side HL. But CB is equal to HG, and the angle BCN is equal to the corresponding outer angle GHL; therefore, the triangles BCN, GHL, are equal (?).

If now the solid FGLE be applied to the solid ABND, so that the face EL shall coincide with its equal DN, then the faces GHL and BCN, being equal and in the same plane, will coincide; and in the same manner it may be shown that the faces FEK and ADM will coincide; hence, also, the face FH will coincide with the face AC, and FL with AN; and the two solids are consequently equal. If, then, from the whole solid AL, we take away these equal portions in turn, the remainder AGHD will be equivalent to the remainder AGLM.

Therefore, parallelopipeds on the same base, etc.

Sch.—Any two parallelopipeds on the same base and between the same planes are equivalent. For, if their upper bases are not between the same parallels, but have some other position, as DH and MS, produce two sides of each till they intersect as in the figure; then, UVWX can be the upper base of a parallelopiped on the same lower base and between the same parallels with both of the other parallelopipeds; hence, by the above dem-

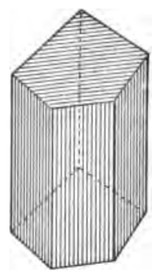


onstration, each of the others will be equivalent to this third parallelopiped; they will, therefore, be equivalent to each other.

Cor.—The volume of any parallelopiped on a rectangular base, is equal to the area of its base multiplied by its altitude: for it is equivalent to a right parallelopiped of the same base and altitude.

DEFINITIONS.

11. A **prism** is a polyedron bounded by plane faces, of which two are equal and parallel polygons, and the others parallelograms. It includes the parallelopiped as one species.



The equal and parallel polygons are called the **bases.** The other faces together form the **convex** or **lateral surface.** The edges joining the corresponding angles of the two bases are called the *principal edges.*

A prism is called triangular, quadrangular, pentagonal, etc., according as its base is a triangle, a quadrilateral, a pentagon, etc.

12. A **right prism** is one whose principal edges are perpendicular to the bases. Any other prism is called **oblique.**

13. A **cylinder** is a solid described by the revolution of a rectangle about one side, which remains fixed.

The fixed side is called the **axis** of the cylinder. The opposite side describes the **convex surface.** The circles described by two other sides are called the **bases.**

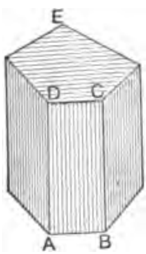
14. A prism is said to be **inscribed** in a

cylinder when its bases are inscribed in the bases of the cylinder, and its principal edges lie in the convex surface of the cylinder.

15. The **height, or altitude,** of a prism or a cylinder, is the perpendicular distance between the planes of its bases.

THEOREM XX.

The convex surface of a right prism is equal to the perimeter of its base multiplied by its height.

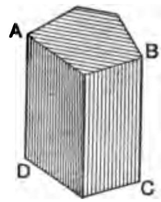


Let ABE be a right prism. Since its principal edges are, by definition, perpendicular to the bases, any one of them may be taken as the height of the prism. Now, AD and BC, being both perpendicular to the lower base, are also perpendicular to AB, which they meet in the plane of that base (?); hence, the parallelogram ABCD is a rectangle, and its area is equal to its base AB multiplied by its altitude BC (Cor. 1, Theo. XXIV, Book I). In the same manner it may be shown that the area of each of the other faces composing the convex surface is equal to its base multiplied by the altitude of the prism. Therefore, the sum of their areas is equal to the sum of their bases multiplied by the common altitude. But the sum of their bases constitutes the perimeter of the base of the prism.

Hence, the convex surface, etc.

THEOREM XXI.

The volume of any prism is equal to the area of its base multiplied by its altitude.

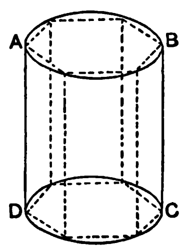


Let ABCD be any prism. Now, whatever may be the form of its base, AB, it is evident that it may be divided into an indefinitely large number of small rectangles, so that the remainder, if there be any, shall be less than any assignable quantity; and the base DC may be divided into the same number of equal rectangles having their sides respectively parallel to the sides of the others. Now, each rectangle in the lower base, and its corresponding rectangle in the upper base, may be considered as the opposite bases of a small parallelopiped; and the prism will be made up of such parallelopipeds. But the volume of each of these will be equal to the area of its base multiplied by its altitude (Cor., Theo. XIX), which is the same as the altitude of the prism; hence, the sum of their volumes will be equal to the sum of their bases multiplied by their common altitude.

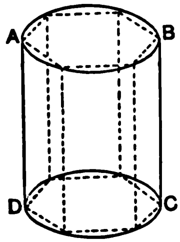
That is, the volume of any prism is equal, etc.

THEOREM XXII.

The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude; and its volume is equal to the area of its base multiplied by its altitude.



Let ABCD be a cylinder, having a prism inscribed in it whose base is a regular polygon. Now, if the number of sides of this polygon be indefinitely increased, its perimeter will ultimately coincide with the circumference of the base of the cylinder. Then, also, the convex surface

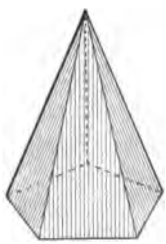


of the prism will coincide with the convex surface of the cylinder, and the volume of the prism with the solidity of the cylinder. But the convex surface of the prism is equal to the perimeter of the base multiplied by the altitude (Theo. XX), and its volume is equal to the area of the base multiplied by the altitude (Theo. XXI).

Therefore, the convex surface of a cylinder, etc.

SECTION XVII.—PYRAMIDS AND CONES.

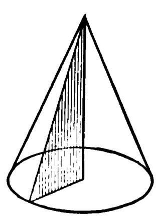
DEFINITIONS.



1. A **pyramid** is a polyedron bounded by plane faces, of which one is any polygon, and the others triangles having a common vertex. The polygon is called the **base**. The triangles together form the **convex** or **lateral** surface.

Pyramids are called triangular, quadrangular, pentagonal, etc., according as their bases are triangles, quadrilaterals, pentagons, etc.

2. A **regular pyramid** is one whose base is a regular polygon, and the triangular faces are equal and isosceles.



3. A **cone** is a solid described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed. The fixed side is called the **axis** of the cone. The hypotenuse describes the **convex surface.** The circle described by the other revolving side is called the **base.**

4. The **altitude** of a pyramid or cone is the perpendicular distance from the vertex to the plane of the base.

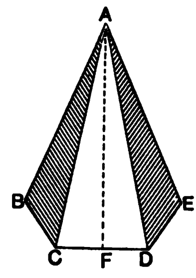
5. The **slant height** of a regular pyramid is the perpendicular let fall from the vertex upon the base of any one of its triangular faces. The **side** or **slant height** of a cone is the straight line drawn from the vertex to any point in the circumference of the base.

6. A **frustum** of a pyramid or cone is the portion next the base cut off by a plane parallel to the base. The slant height of a frustum is that part of the slant height of the whole solid which lies on the frustum.

7. A **section** of any solid is the surface in which it is divided by a plane which passes through it.

THEOREM XXIII.

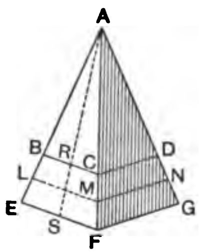
The convex surface of a regular pyramid is equal to half the product of the perimeter of the base by the slant height.



Let A-BCDE be a regular pyramid, of which AF is the slant height. The area of the triangle ACD is equal to half the product of its base CD into its altitude, which is the slant height. Consequently, the areas of all the equal triangles composing the convex surface are together equal to half the product of the sum of their bases by the slant height. But the sum of their bases constitutes the perimeter of the base of the pyramid.

Therefore, the convex surface of a regular pyramid, etc.

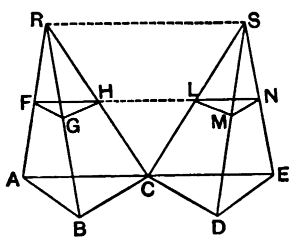
Sch. 1.—The trapezoids BF, CG, etc., composing the convex surface of a frustum of a regular pyramid, are equal to each other, for they are the differences between the equal triangles AEF, AFG, etc., and the equal triangles ABC, ACD, etc.



Sch. 2.—Since the area of the trapezoid BF is equal (Cor. 2, Theo. XV, Book II) to its mean breadth, LM multiplied by RS, which is the slant height of the frustum, and since the same is true of each of the other trapezoids, it follows that *the convex surface of a frustum of a regular pyramid is equal to its slant height multiplied into the perimeter of a middle section between its two bases.*

THEOREM XXIV.

Two triangular pyramids of equal bases and altitudes are equivalent.



Let the two pyramids have their bases in the same plane, and DC equal to BC. Conceive a plane to cut the two solids parallel to the plane of their bases, making the triangular sections FGH and LMN. Now, since the vertices R and S are, by hypothesis, equidistant above the plane of the bases, a third plane may pass through these two points parallel with the other two planes (Cor., Theo. II). Then, the lines HL and RS, being the intersections of two parallel planes by a third plane CRS, will be parallel (Def. 7,

Sec. XIV); hence, the triangles CRS, CHL, are similar (Cor., Theo. XI, Book II). In the same manner it may be shown that the triangles CRB, HRG, are similar; also CSD and LSM.

$$\text{Now, } CR : HR = CB : HG.$$

$$\text{Also, } CS : LS = CD : LM.$$

But, by similarity of CRS and CHL, we have,

$$CR : HR = CS : LS.$$

Hence, by equality of ratios,

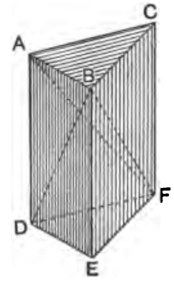
$$CB : HG = CD : LM.$$

But CB is, by hypothesis, equal to CD; therefore, HG is equal to LM. In the same way, it may be shown that the other sides of the triangle FHG are respectively equal to the other sides of the triangle LMN; and these triangles are consequently equal throughout. Hence, at equal heights, the sections parallel to the bases are equal; and the two pyramids may be conceived to be applied to one another so as to coincide at all equal heights successively, from their bases to their vertices. They are, therefore, equivalent.

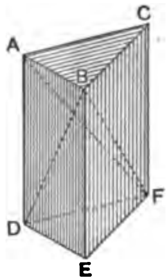
That is, two triangular pyramids, etc.

THEOREM XXV.

A triangular pyramid is one third of a triangular prism of the same base and altitude.



Let ABCDEF be a triangular prism. Join AF, BF, and BD. Now, the pyramid B-DEF, cut off by the plane of the triangle BDF, is equivalent (Theo. XXIV) to the pyramid F-ABC, cut off by the plane of the triangle ABF; for they have equal bases, DEF and ABC (Def. 11, Sec. XVI), and the same altitude, namely, the altitude of the prism.



But the pyramid F-ABC is equivalent to the third pyramid B-ADF; for they have equal bases, ADF and FCA (Cor. 1, Theo. XX, Book I), and the same altitude, namely, the perpendicular distance of their common vertex B above the plane of their bases ADFC. Hence, the pyramid B-DEF is one third of the prism ABCDEF.

That is, a triangular pyramid is one third, etc.

Cor. 1.—Hence, the volume of a triangular pyramid is equal to one third of the product of its base by its altitude (Theo. XXI).

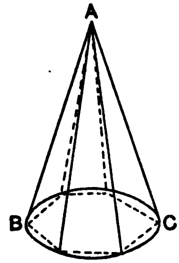
Cor. 2.—The volume of any pyramid whatever, is equal to one third the product of its base by its altitude; for, by dividing its base into triangles, and passing planes through the lines of division and the vertex, the pyramid can be divided into a number of triangular pyramids; and the sum of their volume will be equal to one third the product of the sum of their bases by their common altitude.

THEOREM XXVI.

The convex surface of a cone is equal to half the product of the circumference of the base by the slant height; and its volume is equal to one third the product of the area of the base by the altitude.

Let ABC be a cone, having a regular pyramid inscribed in it. If the number of sides of the polygon constituting the base of the pyramid be indefinitely increased, its perimeter will ultimately coincide with the circumference of the base of the cone. Then, the

slant height of the pyramid will be equal to the slant height of the cone, and the convex surface and solidity of the pyramid to the convex surface and solidity of the cone. But the convex surface of the pyramid will be equal to half the product of the perimeter of the base by the slant height (Theo. XXIII); and the volume of the pyramid will be equal to one third the product of the area of the base by the altitude (Cor. 2, Theo. XXV).



Therefore, the convex surface of a cone, etc.

Sch.—In the same manner, it may be shown that the convex surface of a frustum of a cone is equal to the product of the slant height into the circumference of a middle section between the two bases (Sch. 2, Theo. XXIII). And as this will hold true however small the upper base may be, it will hold true of the cone itself, which may be treated as a frustum whose upper base is nothing.

EXERCISES.

1. If the edge of a cube is x, what is the volume? The surface? The diagonal?

2. The altitude of a right pyramid is a, its base is an equilateral triangle described about a circle whose radius is r. Find the volume and surface.

3. The slant height of a cone is s, and the radius of base r. Find the volume and the entire surface.

4. Prove that the vertex of a regular pyramid (Def. 2, Sec. XVII) is in a line perpendicular to the base at its center.

Ques. 1.—What is true of all the lateral edges of a regular pyramid?

Ques. 2.—What is the locus of all the points equally distant from every point in the circumference of a circle?

Ques. 3.—Why is every regular polygon capable of being inscribed in a circle?

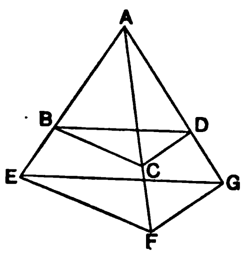
5. The edges of a regular tetraedron are each $2e$. Show that the surface is $4e^2\sqrt{3}$.

6. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of three edges meeting at a common vertex.

7. Two prisms, or pyramids, having equivalent bases, are to each other as their altitudes. Having equal altitudes, they are proportional to their bases.

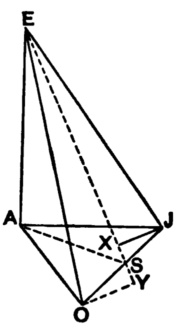
8. The lateral surface of a pyramid or cone is greater than the base.

9. The portion of a tetraedron cut off by a plane parallel to any face is a tetraedron similar to the given one.



If the plane BCD is parallel to EFG, the tetraedron A-BCD is similar (Def. 6, Sec. XVI) to A-EFG (?).

10. Two pyramids are equal when the base and two adjacent sides of each are respectively equal.



11. The plane which bisects a diedral angle of a tetraedron divides the opposite edge in the ratio of the areas of the adjacent faces.

The plane SAE bisects the diedral AE, and so divides the edge OJ that EAO : EAJ $=$ OS : JS.

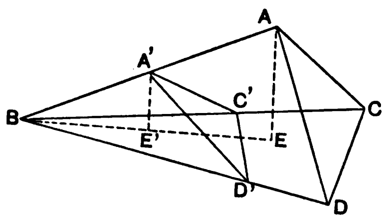
Let fall the perpendiculars JX and OY upon the plane SAE. They are, therefore, parallel, and (1) OY : JX $=$ OS : JS (?).

The tetraedrons O-AES and J-AES have a common base, AES; and, therefore, are to each other as their altitudes. Denoting the tetraedrons by V and V′, and (2) V : V′ = OY : JX = OS : JS.

The tetraedrons S-EAO and S-AEJ have equal altitudes(?). These volumes are V and V′(?). Therefore, (3) V : V′ = EAO : EAJ (?). Complete.

12. Two tetraedrons, which have a triedral angle of the one equal to a triedral angle of the other, are to each other as the products of the three edges of the equal diedrals.

A-BCD and A′-BC′D′ are the tetraedrons having a common triedral, B. Prove that they are to each other as BA × BC × BD is to BA′ × BC′ × BD′.



AE and A′E′ are altitudes.

The bases are to each other as BD × BC is to BD′ × BC′(?).

The altitudes are to each other as BA is to BA′(?).

13. What is the entire surface of a cone whose slant is 50 feet, and whose base is 10 feet in diameter?

14. What is the volume of a cone having half the altitude of the above and double its base?

15. The dimensions of a bushel measure are 18½ inches in diameter and 8 inches deep. What would be the dimensions of a gallon measure of the same shape?

Ques.—In what ratio are similar solids to each other?

16. The altitude of a right pyramid is 2 feet, and the base is a regular hexagon whose sides are each 8 inches. Find the entire surface.

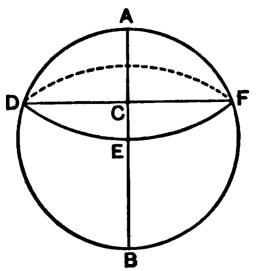
17. What is the edge of a cube equal in volume to the pyramid given in the 16th Ex.?

SECTION XVIII.—THE SPHERE.

DEFINITIONS.

1. A **sphere** is a solid which may be described by the revolution of a semicircle around its diameter as a fixed axis.

The semi-circumference describes the **convex surface**. The **center** is the middle point of the axis. The **radius** is a straight line from the center to any point of the surface; and it is equal to the radius of the semicircle. A **diameter** is a double radius.



Sch.—As the semicircle ADB revolves about the axis AB, a perpendicular, as DC, let fall on the axis, from any point in the circumference, will describe a circle.

Cor. 1.—All points on the surface of a sphere are equally distant from the center(?).

Cor. 2.—Every plane section of a sphere is a circle(?).

Ques. 1.—What shape is the horizon when viewed at sea, or upon any other level extent of surface; and at different distances from the surface?

Ques. 2.—What does this prove as to the shape of the earth?

2. A **great circle** on a sphere is one whose plane passes through the center. Its radius is the same as the radius of the sphere. Its circumference is also called the circumference of the sphere. Any other circle on the sphere is called a **small circle**.

Cor. 1.—All great circles of a sphere are equal(?).

Cor. 2.—Two great circles of a sphere are never parallel (?).

Cor. 3.—Every great circle of a sphere bisects the surface of the sphere (?).

Cor. 4.—Two great circles of a sphere mutually bisect (?).

Ques. 1.—Why does the intersection of the planes of two great circles pass through the center?

Ques. 2.—If it passes through the center, what is it?

Cor. 5.—Of two small circles, the less is the farther from the center (?).

3. A portion of a sphere cut off by a plane, or included between two parallel planes, is called a **segment.**

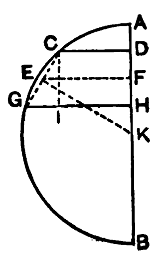
4. The curved surface of a segment is called a **zone.** The altitude of the segment is the altitude of the zone.

Ques.—How is a zone bounded?

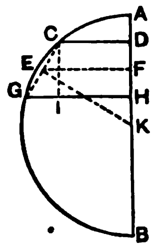
5. A frustum of a cone is said to be **inscribed in a sphere,** when the circumferences of its bases lie in the surface of the sphere.

THEOREM XXVII.

If a frustum of a cone be inscribed in a sphere, its convex surface will be equal to the altitude of the frustum multiplied by the circumference of a circle whose radius is a perpendicular from the center of the sphere to the slant height of the frustum.



Let CD and GH be both perpendicular to the axis AB. Then, as the semicircle revolves about the axis, describing the sphere, the trapezoid DCGH will describe a frustum of a cone, which will be inscribed in the sphere. From the center K let fall the



perpendicular KE, on the chord CG; then E will be the middle point of CG (Theo. XXXIII, Book I). Draw EF perpendicular to AB, and CI perpendicular to GH.

Now, since the triangles GCI, KEF, have the sides of the one respectively perpendicular to the sides of the other, they are similar (Theo. XII, Book II).

Hence, GC : CI = KE : EF.

Multiplying an extreme and a mean equally (Theo. V, Book II.

$$GC : CI = 2KE \times 3.14159 : 2EF \times 3.14159.$$

The first term of this proportion is the slant height of the frustum; the second is its altitude; the third is the circumference of a circle whose radius is KE (Cor. 1, Theo. XXI, Book II); the fourth is the circumference of a circle whose radius is EF. Therefore, multiplying extremes and means, we have the product of the slant height into the circumference of a middle section of the frustum, equal to the product of its altitude into the circumference of a circle whose radius is the perpendicular from the center to the slant height. But the former product is equal to the convex surface of the frustum (Sch., Theo. XXVI); consequently, the latter product is equal to the same.

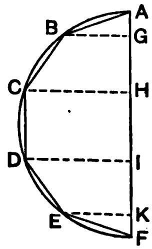
That is, if a frustum of a cone be inscribed, etc.

Sch.—If the chord CG be one side of a regular inscribed polygon, is is evident that KE will be its apothegm (Book I, Theo. XXIX, Def. 2).

Cor.—If a square be rotated upon its diagonal as an axis, the convex surface of the double cone thus generated will be equal to the circumference of the circle inscribed in the square multiplied by the diagonal of the square. Similarly, rotate an octagon, and examine the surface of the figure.

THEOREM XXVIII.

The surface of a sphere is equal to the product of its diameter by its circumference.



Let ABCDEF be a semicircle, having the half of a regular polygon inscribed in it. As the semicircle revolves about the axis AF, describing the sphere, each of the trapezoids GBCH, HCDI, etc., will describe a frustum of a cone, which will be inscribed in the sphere. The convex surface of each of these frustums will be equal to its altitude multiplied by the circumference of a circle whose radius is the apothegm of the polygon (Sch., Theo. XXVII). Therefore, the sum of their convex surfaces will be equal to the circumference of such a circle multiplied by the sum of the altitudes GH, HI, etc.; that is, multiplied by AF, the diameter of the sphere. Now, if the number of sides of the semi-polygon be indefinitely increased, its perimeter will ultimately coincide with the semi-circumference, and its apothegm with the radius, of the sphere. Then, the sum of the convex surfaces of the frustums will be equal to the surface of the sphere, and the circumference of which the apothegm is radius will be the circumference of the sphere (Def. 2).

Therefore, the surface of a sphere is equal, etc.

Cor. 1.—Since the circumference is equal to 3.14159 $\times$ D (Cor. 1, Theo. XXI, Book II), it follows that the surface is equal to $3.14159 \times D^2 = \pi D^2 = 4R^2\pi$. $R^2\pi$ is the area of a circle whose radius is R. Therefore, the surface of a sphere is four times the area of one of its great circles.

Cor. 2.—The area of a zone is equal to the product of its altitude by the circumference of a great circle.

THEOREM XXIX.

The volume of a sphere is equal to its surface multiplied by one sixth of its diameter.

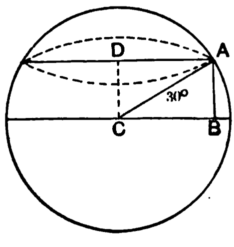
The sphere may be considered as made up of infinitely small pyramids, whose bases together form the surface of the sphere, and whose common vertex is at the center. Now, the volume of each of these pyramids will be equal (Cor. 2, Theo. XXV) to the product of its base by one third of its altitude; that is, by one third the radius of the sphere. Hence, the sum of their volumes will be equal to the sum of their bases multiplied by one third of the radius, or one sixth of the diameter.

That is, the volume of a sphere is equal, etc.

Cor.—Since the surface of a sphere is equal to πD^2, it follows that its volume is equal to $\pi D^2 \times \frac{D}{6}$, or $D^3 \times \frac{\pi}{6} = D^3 \times .5236$.

EXERCISES.

1. Assuming the earth to be a sphere whose radius is 3962.72 miles, what is the hourly rate of rotation of a point on the equator?



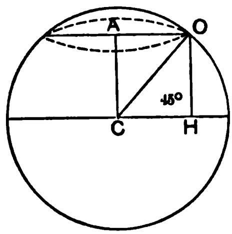
2. What is the hourly rate of a point on the thirtieth parallel (Ex. 5, Sec. V)?

Ques.—The line AB is equal to what part of AC, or radius?

AD = CB, for ABCD is a parallelogram.

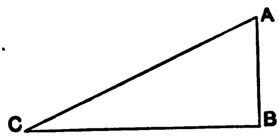
AD is the radius of the parallel circle of 30°.

3. What is the hourly rate of a point on the forty-fifth parallel?



Ques.—The line OH = what part of a radius?

Sch.—In a right-angled triangle, as CAB, the ratio of the base CB to the hypotenuse CA, that is $\frac{CB}{CA}$, is called the **cosine** of the angle C.



So the ratio of a line such as CB, or CH, in Problems 2 and 3, to the radius of the earth, is called the **cosine of the latitude.**

4. The rate of hourly rotation of a point on the earth's surface, and the length in miles of a degree of longitude, each varies with the cosine of the latitude.

5. What is the length in miles of a degree of the thirtieth parallel? Of the forty-fifth? Of the sixtieth?

6. How much of the earth's surface is included between the equator and the thirtieth parallel? Between the thirtieth and the sixtieth? Between the sixtieth and the North Pole?

7. With the same data as before concerning the shape and radius of the earth, the altitude of the North Torrid Zone is 1580.12 miles; of the North Temperate Zone, 2053.67 miles; of the North Frigid Zone, 329 miles. How many square miles in each zone?

8. Conceive a triedral angle at the center of a sphere. Its planes cut the surface in arcs of great circles, which form a *spherical triangle.*

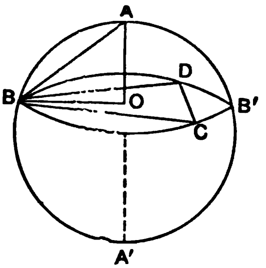
Cor. 1.—Any side of a spherical triangle is less than the sum of the other two (?).

Cor. 2.—The perimeter of a spherical triangle is less than a great circle (?).

9. To find the surface of a material sphere.

Suppose that the given sphere be of wood or iron.

Let $ABA'B'$ be a great circle of a sphere which is to represent the given sphere.



Place one point of the compasses at A and the other at B, and draw the small circle $BCB'D$. The unknown diameter AA' will be divided into two segments, AO and $A'O$; and BO, the radius of the small circle, will be a mean proportional between these segments(?) We must first find BO. With the compasses measure the distances BC, BD, and DC, and construct a triangle on a plane surface with these lines. The circle which circumscribes this triangle is equal to the circle $BCB'D$, and its radius is, of course, equal to BO.

With BO and AB as sides of a right-angled triangle, we can find AO, one segment of the diameter; and as $AO : BO = BO : A'O$, we have the other segment. Half their sum is the radius, and $4R^2\pi =$ the surface.

10. Find the inner surface of a spherical shell whose exterior diameter is 10 inches, and whose thickness is $\frac{1}{2}$ an inch.

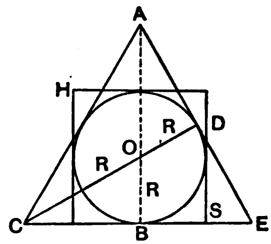
11. Find the surface of a sphere whose volume is 1000 cubic inches.

12. Prove the surface of a cube equal to twice the square of the diameter of the circumscribed sphere. $6S^2 = 2D^2$.

13. The diameter of a sphere is 15 inches. Find the edge of the largest cube that can be cut out of it.

14. Three equal circles touch each other externally, thus inclosing an acre of ground. What is the radius, in rods, of the equal circles?

15. The surface and volume of a sphere are respectively equivalent to two thirds the surface and volume of the circumscribing cylinder. And if a cone whose slant height is equal to the diameter of its base be circumscribed about the sphere, the surface and volume of the cylinder will be respectively equal to two thirds the surface and volume of the cone.



Since $OB = \frac{1}{2}CO$(?), $R = \frac{1}{3}CD$(?). The altitude of the cone is $3R$, and its slant height is $2R\sqrt{3}$(?). Rotate the figure about AB, and the circle generates a sphere; the rectangle HS, a circumscribing cylinder; the equilateral ACE, the circumscribing cone.

Surface of sphere, $4\pi R^2$.
" " cylinder, $6\pi R^2$.
" " cone, $9\pi R^2$.

Volume of sphere, $\frac{4}{3}\pi R^3$.

" " cylinder, $\frac{6}{3}\pi R^3$.

" " cone, $\frac{9}{3}\pi R^3$.

16. The bases of a cone and cylinder are each equal to the great circle of a sphere, and their common altitude is the diameter of the sphere. Prove that the volumes of cone, sphere, and cylinder are as the numbers 1, 2, and 3.

1.00

Elements
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(Webster Wells S.B.
Revised.